



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: February 27, 2016

ACCEPTED: May 30, 2016

PUBLISHED: June 16, 2016

# Witten index for noncompact dynamics

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**ABSTRACT:** Among gauged dynamics motivated by string theory, we find many with gapless asymptotic directions. Although the natural boundary condition for ground states is  $L^2$ , one often turns on chemical potentials or supersymmetric mass terms to regulate the infrared issues, instead, and computes the twisted partition function. We point out how this procedure generically fails to capture physical  $L^2$  Witten index with often misleading results. We also explore how, nevertheless, the Witten index is sometimes intricately embedded in such twisted partition functions. For  $d = 1$  theories with gapless continuum sector from gauge multiplets, such as non-primitive quivers and pure Yang-Mills, a further subtlety exists, leading to fractional expressions. Quite unexpectedly, however, the integral  $L^2$  Witten index can be extracted directly and easily from the twisted partition function of such theories. This phenomenon is tied to the notion of the rational invariant that appears naturally in the wall-crossing formulae, and offers a general mechanism of reading off Witten index directly from the twisted partition function. Along the way, we correct early numerical results for some of  $\mathcal{N} = 4, 8, 16$  pure Yang-Mills quantum mechanics, and count threshold bound states for general gauge groups beyond  $SU(N)$ .

**KEYWORDS:** Extended Supersymmetry, Supersymmetry and Duality, D-branes, M(atr ix) Theories

**ARXIV EPRINT:** [1602.03530](https://arxiv.org/abs/1602.03530)

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## 1 Index vs. twisted partition function

The notion of index of elliptic operators has served theoretical physics in countless problems in supersymmetric context. The relevant object in the latter language is the Witten index [1], defined as a trace over the physical Hilbert space,

$$\mathcal{I} = \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^F \cdots e^{-\beta H} \right],$$

where  $\beta$  is the Euclidean time span,  $H = Q^2$  for some choice of supercharge  $Q$ , and the ellipsis denotes any operators that commute with  $Q$ ,  $H$ , and  $(-1)^F$ . Its key property,

guaranteed by integral nature of the quantity, is the invariance under “small” deformations. Since “small” deformation can only induce a continuous change, an integral quantity like this cannot be affected. Hence, one is free to modify the elliptic operator in question, or alternatively the supersymmetric dynamics in question, in any way as long as the underlying chirality or supersymmetric property is intact. For example, if the dynamics has a gap, one can choose to scale it up to an arbitrarily large value and make the spectrum completely discrete, without affecting the value of the index.

In a more modern form, this type of thinking leads to the so-called localization procedure, where one performs the deformation at the level of path-integral, without having to go to the Hamiltonian formulation. The localization works because of two key facts. The first is an introduction of “chemical potentials” as complex parameters that carry the grading information under global symmetries [2], and the second is the identification of a convenient supercharge which acts much like the old-fashioned BRST operators [3, 4]. The former can introduce various gaps to the index problem and should not be thought of as a mere reservoir of charge information. The latter then allows deformations of the dynamics that scale up the gaps. While these two elements are hardly new as mathematical ideas, systematic institution at the level of path-integral was quite recent, leading to many new computational tools and results.

However, such conveniences can often be taken too far if one fails to appreciate what is meant by “small” deformation. A well-known example of this subtlety can be found in supersymmetric quantum mechanics with four supercharges or less, where wall-crossing phenomena occur. For quantum mechanics associated with  $d = 4$  BPS dyons, such wall-crossing phenomenon [5–9] reflects that of  $d = 4$   $\mathcal{N} = 2, 4$  theories [10–14]. For a gauged quantum mechanics [9], its Fayet-Iliopoulos (FI) constant  $\zeta$ , if present in the theory, can change the Witten index discontinuously, as  $\zeta$  is dialed continuously across the  $\zeta = 0$  wall. While a naive invariance argument seemingly applies to the continuous parameter  $\zeta$ , one can see upon closer inspection that setting  $\zeta = 0$  opens up a new gapless asymptotic direction along the vector multiplet [15]. Therefore,  $\zeta \rightarrow 0$  does not qualify as a “small” deformation.

For most physics problems, in fact, this subtlety is associated with asymptotic flat directions. Whenever we encounter a supersymmetric theory with (partially) continuous spectrum, we must be very careful. When the asymptotic direction is gapped, say at  $E = E_{\text{gap}}$ , this is more of a nuisance than a problem. The path-integral for the index can yield a non-integral expression with an additive piece of type

$$\sim e^{-\beta E_{\text{gap}}},$$

which, since we cannot take  $\beta \rightarrow \infty$  in actual path-integral, can compromise the computation of the integral index. Nevertheless, a deformation that scales up  $E_{\text{gap}} \rightarrow a \cdot E_{\text{gap}}$  with  $a \rightarrow +\infty$  allows us to isolate the integral index relatively easily. On the other hand, if one finds an asymptotic direction with  $E_{\text{gap}} = 0$ , as in  $\zeta = 0$  case discussed above, the subtlety becomes more pronounced.

Well-known examples of the latter are the nonlinear sigma models (NLSM’s) with a noncompact target manifold. While the index itself should be a well-defined and integral object, path-integral computation or alternatively naive application of Atiyah-Singer index

theorem will usually compute the  $\beta \rightarrow 0$  limit of the trace formula, often denoted as  $\mathcal{I}_{\text{bulk}}$ , which is typically non-integral. To compute the true index, one must also figure out the difference of the two limits, at  $\beta \rightarrow 0$  and  $\beta \rightarrow \infty$ ,

$$\delta\mathcal{I} \equiv \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^F \cdots e^{-\beta H} \right] - \lim_{\beta \rightarrow 0} \text{tr} \left[ (-1)^F \cdots e^{-\beta H} \right] .$$

The only known systematic computation of this so-called “defect” term or the boundary term,  $\delta\mathcal{I}$ , is for the Atiyah-Patodi-Singer boundary condition,<sup>1</sup> which is *not* in general equivalent to the physical  $L^2$  boundary condition for the original problem.

A popular alternative that can produce an integral expression, at the risk of modifying the asymptotic dynamics, is to introduce chemical potentials [2]. The index, upon such a modification to the dynamics, is then given as

$$\Omega(\mathbf{y}, x; \beta) \equiv \text{tr} \left[ (-1)^F \mathbf{y}^{R+\cdots} x^{G_F} e^{-\beta H} \right] , \quad (1.1)$$

where  $G_F$  denotes the flavor symmetry generators collectively and  $\mathbf{y}$  is distinguished from the rest of the chemical potentials  $x$ , as its exponent involves an R-symmetry generator  $R$ . Such a twisted partition function is well-defined and may produce an integral expression, as long as there is at least one supercharge  $Q$  that commutes with  $R + \cdots$  and obeys  $Q^2 = H$ . In case no such supercharge exists, we simply drop the  $\mathbf{y}$  term. It should be emphasized that the flavor chemical potentials,  $x$ , are not as innocuous as the expression (1.1) might lead one to think. When  $|x| \neq 1$ , they generate a confining potential to the matter dynamics which, of course, is not in general “small,” and hence, one needs to be very careful about them.

Desirably, one must honestly compute

$$\mathcal{I}(\mathbf{y}) \equiv \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^F \mathbf{y}^{R+\cdots} e^{-\beta H} \right] , \quad (1.2)$$

where the trace is taken over physically acceptable states, including  $L^2$  normalizable bound states. In this note, we will take much care to distinguish  $\Omega$ , the twisted partition function with flavor chemical potentials, from  $\mathcal{I}$ , the Witten index. For theories with *compact* dynamics, or equivalently with fully discrete spectra, the naive deformation arguments work well and we have

$$\mathcal{I}(\mathbf{y}) = \Omega(\mathbf{y}, x; \beta) . \quad (1.3)$$

In particular,  $\beta$ - and  $x$ -independence of the right hand side can be shown rigorously for compact theories [15]. In the presence of *gapless asymptotic* directions, however, the relation (1.3) is no longer true,

$$\mathcal{I}(\mathbf{y}) \neq \Omega(\mathbf{y}, x; \beta) , \quad (1.4)$$

and there is, *a priori*, no reason to expect why the true Witten index  $\mathcal{I}$  can be extracted from  $\Omega$  in any straightforward manner.

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<sup>1</sup>For an accessible review for physicists, see ref. [16].

For one thing,  $x$ -dependence of  $\Omega$  can be seen to survive generically and the desired  $x \rightarrow 1$  limit of  $\Omega$  is often divergent. One may still obtain certain integral information out of  $\Omega$  via its Taylor expansion in  $x$  or in  $1/x$ . While it is tempting to attribute the  $x^0$  part of such an expansion to the flavor-neutral sector of the original index problem, this can be seen to fail miserably in some of the simplest examples. For another, when not all of the asymptotic directions in question can be controlled by  $G_F$ , one typically finds fractional expressions for  $\Omega$ , despite the naive integrality argument. This can be understood from the fact that in such cases the path-integral computation of  $\Omega$  takes  $\beta \rightarrow 0$  limit effectively, leading only to the bulk part of the index.

The main purpose of this note is to explore these two quantities, the twisted partition function  $\Omega$  and the Witten index  $\mathcal{I}$ , for theories with gapless asymptotic directions. We will find that, indeed, the  $L^2$  boundary condition is so different from the one introduced by  $\mathbf{y}$  and  $x$ , and that one computation does not translate to the other in any obvious manner. One folklore in the community is that deformations due to chemical potentials, say  $x$ , will not matter much for the states that are neutral under the relevant symmetry generator. This, upon closer inspection, can be seen to have no justifiable ground, however. The argument may work for classical ground states, completely localized at fixed points, but quantum states have a wavefunction spread. An argument of this type may be defensible if we wish to infer the spectrum at a typical value of  $x$  from the limit of  $\Omega$  at  $\log|x| \rightarrow \pm\infty$ , whereby localized ground state wavefunctions will approach classical ones, but *definitely not* for the spectrum at  $\log|x| \rightarrow 0$ , where a wavefunction can easily become nonnormalizable.

Nevertheless, we will note examples where some information about  $\mathcal{I}$  is embedded in  $\Omega$  in a very subtle manner whereby one can, *a posteriori*, recover  $\mathcal{I}$  from  $\Omega$ . In one set of examples, we find that various cohomologies associated with different boundary conditions can be captured by various flavor-neutral sectors in  $\Omega$ . Note that the procedure to isolate a flavor-neutral part is far from being unique and one is thus led to a multitude of cohomology types. Surprisingly, the  $L^2$  cohomology is embedded in those flavor singlets, as a common intersection of two mutually different answers. In another set of examples,  $\Omega$  proves to be fractional rather than integral, yet  $\Omega$  can be understood as a linear combination of integral  $\mathcal{I}$ 's for several related theories. This allows a simple inversion formula from rational  $\Omega$ 's in favor of integral  $\mathcal{I}$ 's. So far, we have no reason to believe that such a recovery of  $L^2$  Witten indices from twisted partition functions is guaranteed, nor do we see a universal routine for our examples. Nevertheless, the mere fact that  $\mathcal{I}$ 's are somehow embedded in  $\Omega$ 's is very suggestive.

This note is organized as follows. Section 2 will summarize the results from Hori-Kim-Yi (HKY) [15], with some more technical material relegated to appendix A. Section 3 will consider several classes of theories with a noncompact moduli space from chiral multiplets, and explore how  $\Omega$ 's and  $\mathcal{I}$ 's are related, with  $\mathcal{N} = 4$  and  $\mathcal{N} = 8$  supersymmetries. Here we will see how in general  $\Omega$  fails to contain  $\mathcal{I}$  information properly, or in other words how there is no obvious way to extract  $\mathcal{I}$  from  $\Omega$ . Nevertheless, we will encounter interesting examples where various flavor expansions of  $\Omega$  seemingly reproduce alternate, albeit physically irrelevant, cohomologies and how in such examples known  $\mathcal{I}$  is extractable, *a posteriori*. However, we caution the reader that these examples should be taken as

anecdotal rather than exemplary. Section 4 will move on to more subtle cases with a gapless asymptotic direction in vector multiplets, which carry fractional  $\Omega$ 's in the end. Pure Yang-Mills quantum mechanics are prototypes, and we will provide detailed relationships between  $\Omega$ 's and  $\mathcal{I}$ 's for this class of theories, which will lead to new computations and predictions about threshold bound states for  $\mathcal{N} = 4, 8, 16$  quantum mechanics for simple gauge groups beyond  $SU(N)$ . Motivated by this, in section 5, we turn to the notion of the rational invariant, which has appeared universally in various wall-crossing formulae [17–24]. There, we will also explain how such a notion can help us to count the very subtle threshold bound states in nonprimitive quiver theories.

## 2 Localization

Quantum mechanical GLSM's with  $\mathcal{N} = 4$  supersymmetries can be obtained by dimensionally reducing  $d = 4$   $\mathcal{N} = 1$  or  $d = 2$   $\mathcal{N} = (2, 2)$  systems [25]. Such theories can be characterized by the gauge group  $G$  of rank  $r$ , the representations  $\mathcal{R}_a$  for the matter fields  $\Phi_{a=1, \dots, A}$ , the Fayet-Iliopoulos (FI) constants  $\zeta$  (one for each gauge group factor with a  $U(1)$  sector), and the superpotential. For  $d = 2$   $\mathcal{N} = (2, 2)$  theories, the FI constant is naturally paired with a  $\theta$ -angle and becomes complex. For  $d = 1$  counterpart, however, physics is independent of  $\theta$  and  $\zeta$  can effectively be thought of as a real parameter again, so that the physics may change discontinuously across the  $\zeta = 0$  wall, leading to the wall-crossing phenomena. For the localization computation of Witten index, which is the main focus of this work, the superpotential is constrained by  $R$ -charges,  $R_a$ , and other flavor charges,  $F_a$ , but is assumed generic otherwise.

For an  $\mathcal{N} = 4$  GLSM quantum mechanics with hamiltonian  $H$ , we are interested in the twisted partition function [15, 23, 26],

$$\Omega(\mathbf{y}, x; \beta) = \text{tr} \left[ (-1)^{2J_3} \mathbf{y}^{2J_3+R} x^{G_F} e^{-\beta H} \right], \quad (2.1)$$

where  $J_3$  and  $R$ , respectively, denote the third component of the  $SU(2)_R$  generators and the  $U(1)_R$  generator, and the flavor symmetry generators are collectively denoted by  $G_F$ , with their associated chemical potentials  $x$ .<sup>2</sup> We will sometimes label this object  $\Omega$  also by the FI constants,  $\zeta$ , to emphasize that it may only be piece-wise constant in the FI-constant space, possibly exhibiting a wall-crossing behavior. In this section, a self-contained explanation will be given of how one can compute  $\Omega$  for a simple case. We provide in appendix A.1 some technical details needed to deal with a general case and refer the readers to ref. [15] for the derivation.<sup>3</sup>

Let us recall that, when the low energy theory is entirely geometric and compact, this is supposed to capture the following Hirzebruch-type index [29],<sup>4</sup>

$$\mathcal{I}_{\text{Hirzebruch}}(\mathbf{y}) = \sum_{p=0}^d \sum_{q=0}^d (-1)^{p+q-d} \mathbf{y}^{2p-d} h^{(p,q)}, \quad (2.2)$$

<sup>2</sup>Normalization of  $R$  is such that the superpotential must have charge 2 to be consistent with this  $R$ -symmetry.

<sup>3</sup>See also refs. [27, 28] for related discussions.

<sup>4</sup>This expression differs from the one used in mathematics literature by the choice of the variable  $\mathbf{y}$  that encodes the grading, and has an additional overall factor  $(-\mathbf{y})^{-d}$ .

of the target manifold, where  $h^{(p,q)}$  are the Hodge numbers and  $d$  is the complex dimension. For such theories,  $\Omega(\mathbf{y}, x; \beta)$  reduces to  $\mathcal{I}_{\text{Hirzebruch}}(\mathbf{y})$  and is in particular independent of  $x$  and  $\beta$ . The overall sign is shifted, relative to the standard mathematics convention, so that the middle cohomology contributes with  $+$  sign; this feature is built into the localization formulae we present below in the sign convention for the fermion one-loop determinants.

Via supersymmetric localization, the path integral for the twisted partition function leads to [15]

$$\Omega^\zeta = \frac{1}{|W|} \text{JK-Res}_\eta^{\text{internal}} g(u) d^r u + \frac{1}{|W|} \text{JK-Res}_{\eta; \zeta}^{\text{asymptotic}} g(u) d^r u, \quad (2.3)$$

whose JK-residue [32] operation is summarized below. In the formula,  $W$  is the Weyl group of  $G$  and  $u$  collectively denotes the zero modes of the Cartan vector multiplets, each of which takes values in a cylinder  $\mathbb{C}^*$  of periodicity  $2\pi i$ . The quantities on the right hand side require some further explanations. Let us briefly clarify what this formula means.

Firstly, the integrand  $g(u)$  takes the form,

$$g(u) = g_{\text{vector}}(u) g_{\text{matter}}(u), \quad (2.4)$$

with the two factors coming from the one-loop determinants of the non-zero modes for the vector multiplets and those for the matter multiplets, respectively. The vector multiplets contribute to the integrand as

$$g_{\text{vector}}(u) = \left( \frac{1}{2 \sinh(\frac{z}{2})} \right)^r \prod_{\alpha \in \Delta_G} \frac{\sinh(-\frac{\alpha \cdot u}{2})}{\sinh(\frac{\alpha \cdot u - z}{2})}, \quad (2.5)$$

where  $z$  is defined by  $e^{z/2} = \mathbf{y}$  and  $\Delta_G$  is the set of root vectors of  $G$ . The matter multiplets in turn contribute as

$$g_{\text{matter}}(u) = \prod_{a=1}^A \prod_{\rho \in \mathcal{R}_a} \frac{\sinh\left(-\frac{\rho \cdot u + (\frac{R_a}{2} - 1)z + F_a \cdot \mu}{2}\right)}{\sinh\left(\frac{\rho \cdot u + \frac{R_a}{2}z + F_a \cdot \mu}{2}\right)}, \quad (2.6)$$

with the flavor chemical potentials  $x = e^\mu$ , where the second product for each matter field  $\Phi_a$  is over the weights  $\rho$  of the representation  $\mathcal{R}_a$ .

For the JK-Res operation in eq. (2.3), one starts out by selecting an arbitrary vector  $\eta$  of length  $r$  [15, 30, 31], the choice of which does not change the final answer as long as  $\eta$  cannot be written as a linear combination of less than  $r$  charges. The latter condition is called “genericity.” One then proceeds to obtain all the co-dimension  $r$  singularities of the integrand  $g(u)$  in the  $u$ -space,  $(\mathbb{C}^*)^r$ . The coordinates of such a singularity will be denoted collectively as  $u_*$  and the collection of charges responsible for  $u_*$  as  $\mathbf{Q}_{u_*}$ . The JK-Res operation is then given as the summation over the singularities of the corresponding JK residues, of which definition we turn to now. One should not forget that there could be contributions from co-dimension  $r - 1$  singularities that extend out to the asymptotic infinity of the  $u$ -space, to which we will come back momentarily.



For notational simplicity, given a singularity  $u_*$ , let us perform a constant shift in  $u$  variables, so that the singularity is located at the origin. Upon such a shift, let us collect all the arguments of the hyperbolic sine functions of the form,  $Q_{i_p} \cdot u$ , from the denominator of the integrand. Then, for the singularity  $u = \mathbf{0}$  at the origin, we have  $\mathbf{Q}_0 = \{Q_{i_p}\}$  in the aforementioned notation, with each charge in  $\mathbf{Q}_0$  contributing a linear vanishing to the denominator. Let us now assume that the origin is a non-degenerate singularity, i.e., the denominator of the integrand gives rise to exactly  $r$  colliding hyperplanes  $Q_{i_p} \cdot u = 0$ , for  $p = 1, \dots, r$ . Then, the JK residue at the origin is defined as the unique linear functional satisfying the following properties [32, 48],

$$\text{JK-Res}_{u=0}^{\eta; \mathbf{Q}_0} \frac{d^r u}{\prod_{p=1}^r Q_{i_p} \cdot u} \equiv \begin{cases} \frac{1}{|\det(Q_{i_1} \dots Q_{i_r})|} & \text{if } \eta \in \text{Cone}(Q_{i_1}, \dots, Q_{i_r}) \\ 0 & \text{otherwise,} \end{cases} \quad (2.7)$$

where “Cone” denotes the cone given by positive spans of the specified charge vectors. In the end, this contributes to the internal part of the JK-Res operation,

$$\text{JK-Res}_{\eta}^{\text{internal}} g(u) d^r u = \sum_{u_*} \text{JK-Res}_{u=u_*}^{\eta; \mathbf{Q}_{u_*}} g(u) d^r u, \quad (2.8)$$

where the summation is over all the singularities  $u_*$  of co-dimension  $r$ .

However, the fact that the  $u$  variables live in  $(\mathbb{C}^*)^r$  implies that there could be residues associated with co-dimension  $r - 1$  singularities that extend out to the asymptotic region of  $(\mathbb{C}^*)^r$ . The contributions from these asymptotic singularities are quite subtle and have been worked out in ref. [15] in much detail. The second term in eq. (2.3) takes those into account as,

$$\text{JK-Res}_{\eta; \zeta}^{\text{asymptotic}} g(u) d^r u = \sum_{l_*} \text{JK-Res}_{l_* \cap \{\infty\}}^{\eta; \mathbf{Q}_{l_*} \cup \{Q_{\infty} = -\zeta\}} g(u) d^r u, \quad (2.9)$$

where the summation is over all the complex lines  $l_*$ , given by the intersection of the singularity hyperplanes for  $g(u)$ , and the JK residues are taken at the intersections of  $l_*$  and the asymptotic infinity, denoted as  $\{\infty\}$ . The symbol  $\mathbf{Q}_{l_*}$  represents the collection of charges, whose associated singular loci contain the line  $l_*$  of singularities. Note that, whenever available,  $-\zeta$  is included as if it is the charge vector associated with the asymptotic pole, for the purpose of the JK positivity test.

It may happen that  $\zeta$  does not exist in the given theory, on the other hand, in which case the asymptotic contribution requires a more careful treatment. However, when  $\zeta$  does exist and when we can find  $\eta = \zeta + \delta$  such that it is generic and belongs to the same chamber<sup>5</sup> as  $\zeta$  in the charge space, the asymptotic singularities can be seen to fail the JK-positivity test automatically. This allows us to simplify the formula to [15]

$$\Omega^{\zeta} = \frac{1}{|W|} \text{JK-Res}_{\zeta + \delta}^{\text{internal}} g(u) d^r u. \quad (2.10)$$

<sup>5</sup>The  $r$  dimensional vector space of charges may be divided into chambers where the dividing walls are linear spans of  $r - 1$  collections of linearly independent charge subsets.



There are also other circumstances when contributions from the boundary cancel among themselves completely, regardless of  $\eta$ . No matter what  $\eta$  is chosen, we have

$$\Omega^\zeta = \frac{1}{|W|} \text{JK-Res}_\eta^{\text{internal}} g(u) \, d^r u, \quad (2.11)$$

in such cases. Pure Yang-Mills theories with a simple gauge group, which carries no  $\zeta$  to begin with, belong to this class, as outlined in appendix A.2.

There are further subtleties to be discussed. First, singularities can be of a degenerate type, in which case the number of colliding hyperplanes exceeds the rank  $r$  of the gauge group. In fact, this happens very generically, including bulk of examples in this note. In such situations, a constructive definition of the JK residue can be used [47, 48], which, for the case of non-degenerate singularities, turns out to be equivalent to the above procedure. Examples in this note tend to have highly degenerate singularities, for which we devote appendix A.1 to describe the actual routine we used.

Second, the JK-residue formulation of  $\Omega$  can only be trusted when the relevant charge sets  $\mathbf{Q}_{u_*}$  are projective. The projectivity means that the charges in question can be considered all “positive” with respect to some ordering in the charge space, or equivalently, that they all belong to a half-space. Unfortunately, we may in general, encounter a degenerate singularity  $u_*$  for which  $\mathbf{Q}_{u_*}$  is non-projective. In fact, this happens quite frequently in non-Abelian theories. The derivation outlined in HKY no longer works for such singularities, and no systematic mechanism to deal with them is currently available. The simplest way out would be to deform the pole locations a little so that  $u_*$  will split into several projective ones. This strategy was successfully used in ref. [33], where the desired results anticipated on physical ground were correctly reproduced. We would take this strategy as well if needed, although, in all of our explicit examples, such nonprojective poles did not carry any iterated residue and were thus harmless.

Finally, although we have assumed  $\mathcal{N} = 4$  in phrasing the localization prescription, other types of theories, as long as  $\mathcal{N} \geq 2$ , can be dealt with in a similar fashion. HKY in particular was phrased for  $\mathcal{N} = 2$  to begin with, while  $\mathcal{N} = 8, 16$  is a matter of adding more chiral multiplets. For the latter, we also need to turn on chemical potentials that are consistent with the right superpotential.

### 3 $L^2$ index vs. twisted partition function with chemical potential

In the previous section, we have reviewed how twisted partition function is computed in terms of JK residues via localization. For this to capture the correct Witten index, the dynamics in question must obey several conditions, most constraining of which is that there are no gapless asymptotic directions. However, many physically interesting cases fail to meet this condition.

When the flat direction has a gap, the problem is relatively innocuous. As mentioned earlier, the twisted partition function computation typically yields contributions that scale like

$$e^{-\beta E_{\text{gap}}},$$

which can be removed by modifying the dynamics such that  $E_{\text{gap}} \rightarrow \infty$ . This procedure does not affect the true index of the dynamics as the above contribution comes from the continuum sector at  $E \geq E_{\text{gap}}$ . The JK residue formulae presented in the previous section follows only after taking such a deformation of the theory for the purpose of index computation [15].

On the other hand, when the flat direction has no gap,  $E_{\text{gap}} = 0$ , we must be more careful. The right thing to do is to impose  $L^2$  boundary condition and compute the index, but this is easier said than done. From many works in the recent literature, including those that lead to the JK residue formulae of the previous section, the “index” is computed with chemical potentials inserted, which introduces massive deformations that lift the flat directions. For the cases where the original dynamics is compact to begin with, this merely leads to the equivariant version of the index and allows a fast and universal computation. When the original dynamics contains gapless asymptotic direction, however, we must take more care since the infrared regulator thus introduced may not be consistent with the natural  $L^2$  boundary condition.

In this section, we consider how chemical potential deals with a gapless asymptotic direction from chiral multiplets. For  $d = 1$ , in the presence of a gapless Coulombic flat direction at quantum level, introduction of chemical potential is not quite enough to control the infrared issue. This is because a gauge multiplet in  $d = 1$  comes with three noncompact scalars at classical level, instead of two as in  $d = 2$ . Usual index computation leads to fractional coefficient, signaling contributions from the continuum sector. Since such cases are qualitatively different from those with a gapless chiral flat direction, we will explore and deal with them separately in section 4 and thereafter.

### 3.1 Flavor expansions of twisted partition functions

Perhaps the simplest yet instructive example is the free theory with a single chiral multiplet with its bosonic degrees of freedom parameterizing a single copy of  $\mathbb{C}$ . Turning on the flavor chemical potential  $x$  associated with  $U(1)$  rotation of the complex plane as the infrared regulator, the path integral gives

$$\Omega^{\mathbb{C}} = \frac{x^{1/2} \mathbf{y}^{-1} - x^{-1/2} \mathbf{y}}{x^{1/2} - x^{-1/2}}. \quad (3.1)$$

Depending on the sign of  $\log |x|$ , one gets two different expansions in  $1/x$  or  $x$ . The former, for example, is

$$\Omega^{\mathbb{C}} = \mathbf{y}^{-1} + (\mathbf{y}^{-1} - \mathbf{y}) \cdot (x^{-1} + x^{-2} + \cdots), \quad (3.2)$$

while the latter has the same form except  $x \rightarrow 1/x$  and  $\mathbf{y} \rightarrow 1/\mathbf{y}$ ,

$$\Omega^{\mathbb{C}} = \mathbf{y} + (\mathbf{y} - \mathbf{y}^{-1}) \cdot (x + x^2 + \cdots). \quad (3.3)$$

Note that the infinite towers, apparent in the expansion with respect to  $x$  or to  $1/x$ , cannot have a physical meaning in view of the original free theory. They are the artifacts of the

confining potential introduced by the chemical potential. One could hope that the flavor-neutral states, at the bottom of each series, make sense physically. The latter suggests a single  $U(1)$  invariant ground state, but note that the respective  $R$ -charges disagree.

There exist cohomologies that are seemingly captured by these results. Recall that the de Rham cohomology, without any boundary condition, is

$$H_{\text{dR}}^n(\mathbb{C}) = \begin{cases} 0 & n = 2 \\ 0 & n = 1 \\ \mathbb{R} & n = 0 \end{cases} \quad (3.4)$$

while the de Rham cohomology with compact support is

$$H_{\text{compact dR}}^n(\mathbb{C}) = \begin{cases} \mathbb{R} & n = 2 \\ 0 & n = 1 \\ 0 & n = 0 \end{cases} \quad (3.5)$$

Recalling how the twisted partition function would be related to the Hirzebruch index if the low energy theory were compact, we see that the flavor-neutral part of eq. (3.2) captures  $H_{\text{dR}}^\bullet(\mathbb{C})$ , while the other expansion (3.3) reflects  $H_{\text{compact dR}}^\bullet(\mathbb{C})$ .

Emergence of the two different cohomologies, depending on the sign of  $\log|x|$ , tells us that something goes wrong when one tries to turn off the chemical potential  $x \rightarrow 1$  for the original gapless dynamics. Furthermore, if one is interested in actual bound state wavefunction, it has to be an  $L^2$  normalizable harmonic form, and no such bound state exists for free theory with the target  $\mathbb{C}$ . Neither of the above two expansions, or the two related cohomologies, counts  $L^2$  harmonic forms by itself, in particular.

A slightly more informative example, complete with a gauge multiplet, is an Abelian GLSM with charges of both signs appearing in matter fields, in the absence of superpotential. Consider an  $\mathcal{N} = 4$   $U(1)$  theory with  $N$   $(+1)$ -charged chirals and  $K$   $(-1)$ -charged chirals. We assign vanishing  $R$ -charges for both sets of chirals so that superpotential is disallowed. With this, the classical moduli space is contractible to either  $\mathbb{P}^{N-1}$  or  $\mathbb{P}^{K-1}$ . As is customary, let us control the noncompact directions along the fibers by introducing chemical potentials. Of  $SU(N) \times SU(K) \times U(1)$  flavor symmetry, it suffices to introduce a single chemical potential  $x$  for the  $U(1)$  flavor under which all charged fields has  $+1$  charge. For example, with  $N = 3$  and  $K = 2$ , we find [15]

$$\begin{aligned} \Omega_{N=3;K=2}^{\zeta>0} = & \left[ (1 + 2x^2)\mathbf{y}^{-4} + (1 - 10x - 3x^2)\mathbf{y}^{-2} \right. \\ & + (1 - 4x^2 + 24x^4 - 4x^6 + x^8) + (-3x^4 - 10x^6 + x^8)\mathbf{y}^2 \\ & \left. + (2x^6 + x^8)\mathbf{y}^4 \right] / (1 - x^2)^4. \end{aligned} \quad (3.6)$$

For general  $N$  and  $K$ , upon expanding  $\Omega$  in either  $x$  or  $1/x$ , we find

$$\Omega_{N;K}^{\zeta>0} = \begin{cases} (-1)^{N+K-1} (\mathbf{y}^{1-N-K} + \dots + \mathbf{y}^{N-K-3} + \mathbf{y}^{N-K-1}) + O(x^2) \\ (-1)^{N+K-1} (\mathbf{y}^{1+K-N} + \mathbf{y}^{3+K-N} \dots + \mathbf{y}^{N+K-1}) + O(1/x^2), \end{cases} \quad (3.7)$$

and the expansion for  $\zeta < 0$  case is obtained by exchanging the roles of  $N$  and  $K$ .

Again, one possible interpretation of these results is that these two flavor-neutral parts of the twisted partition function count, respectively, the de Rham cohomology and the compact-support cohomology, of the moduli space. The de Rham cohomology is homotopy invariant, so that we only need to know  $H^\bullet(\mathbb{P}^{N-1})$  for  $\zeta > 0$ ,

$$H_{\text{dR}}^n(\zeta > 0) = \begin{cases} 0 & n = 2k, N \leq k \leq N + K - 1 \\ \mathbb{R} & n = 2k, k < N \\ 0 & n = 2k + 1, \end{cases} \quad (3.8)$$

while the de Rham cohomology with compact support is obtained from this by Poincare duality as

$$H_{\text{compact dR}}^n(\zeta > 0) = \begin{cases} \mathbb{R} & n = 2k, K \leq k \leq N + K - 1 \\ 0 & n = 2k, k < K \\ 0 & n = 2k + 1. \end{cases} \quad (3.9)$$

The Betti numbers are entirely represented by  $h^{(p,p)}$ , so the leading flavor-neutral parts of the two expansions clearly capture these two sets of cohomologies. As we elaborate below, however, neither counts  $L^2$  harmonic forms.

Exactly why these cohomologies are embedded in  $\Omega$ 's is mysterious. One might be tempted to argue that, at least for the compact cohomology, this is plausible since chemical potential deforms the dynamics grossly at the asymptotic region. The latter is likely to affect less those flavor-neutral wavefunctions with compact support, if there exist any such. For ordinary de Rham cohomology without boundary condition, however, this becomes even less clear. In fact, we will also find examples, later on, where this appearance of various cohomologies is not a universal phenomenon.

### 3.2 Recovering $L^2$ witten index?

A priori, the twisted partition function,  $\Omega$ , cannot be argued to compute physical Witten index for theories with asymptotically flat directions, since the chemical potential introduces a rather drastic modification to the asymptotic dynamics. Nevertheless, one could hope that at least flavor-neutral sectors, often argued to be less sensitive to the chemical potential, might be captured by  $\Omega$ . Recall that, for compact theories, one can rigorously argue that the supersymmetric ground states are all flavor neutral. However, we learned from the previous examples that this rosy picture is nowhere near justified. The notion of flavor-neutral part in  $\Omega$  may crucially depend on the chemical potentials, to begin with, and it does not necessarily agree with what the true Index would have computed.

In these examples, yet, one notices a curious and encouraging fact: although neither of the two Laurent expansions gives us the correct  $L^2$  index for flavor-neutral states, the common intersection of the two expansions does. In other words, if we keep only the flavor-neutral sectors, and then further restrict to those states that appear in the both expansions, this common set is exactly in one-to-one correspondence with the anticipated  $L^2$  states. For the example of  $\mathbb{C}$ , this is trivially so, with no  $L^2$  harmonic forms on  $\mathbb{C} = \mathbb{R}^2$  and no common set in the two expansions for small  $x$  and for small  $1/x$ .

$L^2$  harmonic forms  $\Psi$  admit nondegenerate pairing,

$$\int \Psi \wedge \Psi',$$

and the spectrum must be invariant under  $\Psi \rightarrow *\Psi$ . This is clearly not the case for the cohomologies with or without compact support. However, if we carve out the common intersection of the two cohomologies, Poincare dual to each other, we do obtain a spectrum that is invariant. More precisely, for asymptotically conical geometry, there is a mathematical result due to Hausel et al. [34], which states<sup>6</sup>

$$H_{L^2}^n(M) = \begin{cases} H^n(M, \partial M) & n < d = (\dim_{\mathbb{R}} M)/2 \\ \text{Im}(H^n(M, \partial M) \rightarrow H^n(M)) & n = d \\ H^n(M) & n > d. \end{cases} \quad (3.10)$$

With  $H^k(M, \partial M) = H_{2d-k}(M)$ , we see immediately that  $H_{L^2}$  is mapped to itself under the Poincare map.

For the second example,  $H^n(M) = H_{dR}^n(M)$  and  $H^n(M, \partial M) = H_{2d-n}(M) = H_{\text{compact dR}}^n(M)$  [35], so that

$$\mathcal{I}_{N;K}^{\zeta>0}(\mathbf{y}) \Big|_{L^2} = \begin{cases} (-1)^{N+K-1} (\mathbf{y}^{1+K-N} + \dots + \mathbf{y}^{N-K-1}) & N > K \\ 0 & N \leq K. \end{cases} \quad (3.11)$$

As we already noted, this  $L^2$  Witten index does not coincide with the flavor neutral part of the Index with chemical potential, but, nevertheless, can be extracted by inspecting the two expansions near  $x = 0$  and near  $1/x = 0$ , and isolating the common sector thereof.

Encouraged by this, let us consider a slightly more involved example of  $A_k$  type ALE spaces. These are blowup of  $\mathbb{C}^2/Z_{k+1}$  where  $Z_k$  is embedded in holomorphic  $SU(2)$  isometry of  $\mathbb{C}^2$ , with  $k$  two-cycles parameterized by the  $k$  blowup parameters. While these spaces are hyperKähler, there is a simpler  $\mathcal{N} = 4$  GLSM realization with a choice of preferred complex structure. For this, we start with a  $U(1)^k$  gauge group with  $k+2$  chiral multiplets of gauge charges,

$$\begin{pmatrix} 1, & -2, & 1, & 0, & 0, & 0, & \dots, & 0, \\ 0, & 1, & -2, & 1, & 0, & 0, & \dots, & 0, \\ & & \vdots & & \vdots & & & \\ 0, & \dots, & 0, & 0, & 0, & 1, & -2, & 1, \end{pmatrix}, \quad (3.12)$$

where each line represents the gauge charges of the  $k+2$  chirals with respect to a single gauge  $U(1)$ . We obtain the resolved  $A_k$  ALE space by turning on  $k$  positive FI constants. Since the asymptotic geometry is conical  $\mathbb{C}^2/Z_k$ , flavor chemical potential is needed. Only

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<sup>6</sup>PY is indebted to Edward Witten for independently suggesting a possibility of  $L^2$  cohomologies realized as intersection of  $H_{dR}$  and  $H_{\text{compact dR}}$ ; this is effectively the content of the theorem.

two U(1) flavor symmetries are effective, which we can choose to be represented by the two flavor charge vectors,

$$\begin{aligned} & (1, 0, 0, 0, \dots, 0, 0, 1), \\ & (1, 0, 0, 0, \dots, 0, 0, -1). \end{aligned} \quad (3.13)$$

Similarly, as in the previous example, let us turn on the chemical potential  $x$  associated with the first of these two. Computation of the twisted partition function gives

$$\Omega(\mathbf{y}, x)_{A_k}^{\zeta_i > 0} = \frac{(1 + x^{k+1})(x^2 \mathbf{y}^{-2} + \mathbf{y}^2) + (k(1 + x^{k+3}) - (k+2)x^2(1 + x^{k-1}))}{(-1 + x^2)(-1 + x^{k+1})}, \quad (3.14)$$

from which one quickly realizes that the expansion with respect to  $x$  or  $1/x$  yields, respectively,

$$\begin{aligned} \Omega(\mathbf{y}, x)_{A_k}^{\zeta_i > 0} &= k + \mathbf{y}^2 + O(x), \\ \Omega(\mathbf{y}, x)_{A_k}^{\zeta_i > 0} &= \mathbf{y}^{-2} + k + O(1/x), \end{aligned} \quad (3.15)$$

which suggest, following the same line of thought as above, the common coefficient to  $\mathbf{y}^0$ , i.e.  $k$ , as the true Witten index. This indeed reproduces the well-known  $L^2$  spectrum of  $M_{A_k}$ : it is not difficult to see that the flavor-neutral sectors of these two expansions capture the two different cohomologies of  $A_k$  space,

$$H_{\text{dR}}^n(M_{A_k}) = H^n(M_{A_k}) = \begin{cases} \mathbb{R} & n = 0, \\ \mathbb{R}^k & n = 2, \\ 0 & \text{otherwise,} \end{cases} \quad (3.16)$$

and

$$H_{\text{compact dR}}^n(M_{A_k}) = H^n(M_{A_k}, \partial M_{A_k}) = \begin{cases} \mathbb{R}^k & n = 2, \\ \mathbb{R} & n = 4, \\ 0 & \text{otherwise,} \end{cases} \quad (3.17)$$

respectively, while the  $L^2$  cohomology is, either from eq. (3.10) or from past experiences with these spaces in string theory,

$$H_{L^2}^n(M_{A_k}) = \begin{cases} \mathbb{R}^k & n = 2, \\ 0 & \text{otherwise.} \end{cases} \quad (3.18)$$

Again, this cohomology with physical boundary condition is embedded in  $\Omega$  as the common sector between the two expansions (3.15) with respect to  $x$  and  $1/x$ , just as in the previous rank-one example.

However, note that we have forgotten about the other U(1) flavor symmetries, represented by the chiral charges  $(1, 0, 0, \dots, 0, -1)$ . Using its chemical potential instead, say,  $x'$ , and taking the limits thereof, we find a different pattern,

$$\Omega(\mathbf{y}, x')_{A_k}^{\zeta_i > 0} = k + 1 + O(x') = k + 1 + O(1/x'). \quad (3.19)$$

Following the same prescription as the case of  $x$  would lead to a wrong answer of  $k + 1$ . The choice of chemical potentials to be used as infrared regulator is thus also important, if one wishes to learn about  $L^2$  state counting. Turning on both  $x$  and  $x'$ , this subtlety translates to an order of limit issue. This type of subtleties is quite generic, making the  $L^2$  index extraction out of  $\Omega$  more of an art. For this particular example, the symmetry associated with  $x$  is the one  $U(1)$  isometry that is present for generic  $A_n$  manifold, and the one with  $x'$  is accidental feature of using  $\mathcal{N} = 4$  GLSM realization. Whether this has a particular meaning in the above context is not clear.

Furthermore, these examples so far can hardly be argued to be representative of most general gauged quantum mechanics with gapless asymptotic directions. The former are all Abelian, with rather simple gauge charges. Yet, the manner in which the  $L^2$  cohomology information is embedded in  $\Omega$  is rather intriguing and deserves to be explored further. Some of the obvious questions that need to be answered next are,

*Q1: is there any further simplification if we demand larger supersymmetry?*

*Q2: does such a simple relation between true index  $\mathcal{I}_{L^2}(\mathbf{y})$  and twisted partition function  $\Omega(\mathbf{y}, x)$  persist in more general theories?*

It turns out that a useful laboratory for addressing these questions is the  $\mathcal{N} = 8$  system of ADHM, to which we turn next.

### 3.3 $\mathcal{N} = 8$ and $U(1)$ ADHM

Let us consider  $\mathcal{N} = 8$  GLSM in general. Each hypermultiplet consists of a pair of  $\mathcal{N} = 4$  chirals,  $(H_1^f, H_2^f)$ , in mutually conjugate gauge representations while the  $\mathcal{N} = 4$  vector multiplet is augmented by an adjoint chiral multiplet  $\Phi$  [36]. The complex part of the triplet of  $D$ -terms can be regarded as superpotential that has the universal form

$$\sum_f H_1^f \Phi H_2^f .$$

In addition to the  $U(1)_R$  of  $\mathcal{N} = 4$  subalgebra, which we can choose such that

$$R_\Phi = 2, \quad R_{H_1^f} = R_{H_2^f} = 0,$$

there are two different types of  $\mathcal{N} = 4$  flavor symmetries. One is the universal symmetry,  $\tilde{R}$ , that rotates  $\Phi$  as well as all the  $H$ 's with the charges

$$\tilde{R}_\Phi = -2, \quad \tilde{R}_{H_1^f} = \tilde{R}_{H_2^f} = 1,$$

which appears as a flavor symmetry in  $\mathcal{N} = 4$  sense but really is part of  $\mathcal{N} = 8$  superalgebra. Let us denote its chemical potential as  $\mathbf{z} = e^{w/2}$ . The other type of symmetries,  $\tilde{F}^f$ , only rotates the  $f$ -th hypermultiplet,  $(H_1^f, H_2^f)$ , with

$$(\tilde{F}_{H_1^f}^f, \tilde{F}_{H_2^f}^f) = (1, -1),$$

while  $\tilde{F}_{H_i^{f' \neq f}}^f = 0$  and  $\tilde{F}_\Phi = 0$ . Thus, they are truly a flavor symmetry of  $\mathcal{N} = 8$  theories.



Denoting by  $\tilde{\mu}$  the chemical potential for  $\tilde{F}$ 's collectively, note that  $\Omega$  is always an even function of  $\tilde{\mu}$ . In the integrand  $g$  of the residue formulae, contributions from  $H_1^f$  and  $H_2^f$  are identical to each other except  $\tilde{\mu}^f \rightarrow -\tilde{\mu}^f$ . This means that the two expansions of  $\Omega(\mathbf{y}, \mathbf{z}, \tilde{x})$  under  $\tilde{x} = e^{\tilde{\mu}}$  and under  $1/\tilde{x} = e^{-\tilde{\mu}}$  are necessarily identical and in particular share the identical  $\tilde{F}$ -neutral sector, captured by

$$\tilde{\Omega}(\mathbf{y}, \mathbf{z}) \equiv \Omega(\mathbf{y}, \mathbf{z}, \tilde{x}) \Big|_{\tilde{x} \text{ flavor neutral sector}}. \quad (3.20)$$

Although we will mostly need to turn on the flavor chemical potentials associated with  $U(1)$  factors only, rather than the Cartan part of a non-Abelian flavor symmetry, this observation applies to all such  $\mathcal{N} = 8$  flavor symmetries. If the non-Abelian flavor chemical potentials are used, however, we should be more careful in extracting the singlet part by carefully organizing the expression via characters.

For  $\tilde{R}$ , however, there is no reason to expect the same simplification despite the higher supersymmetry. As with generic flavor chemical potential  $x$  of  $\mathcal{N} = 4$  cases, expansions about  $\mathbf{z} = 0$  and  $1/\mathbf{z} = 0$  would in general disagree. If the same pattern persists, we might expect to find the result to encode  $\mathcal{I}$  of the low energy manifold from careful inspection of the limiting expressions,

$$\tilde{\Omega}(\mathbf{y}, \mathbf{z}) \Big|_{\mathbf{z}^{\pm 1} \rightarrow 0}. \quad (3.21)$$

Note that the choice of  $\mathbf{y}$  as the preferred  $R$ -symmetry chemical potential that carries the cohomology grading is in principle ambiguous, since it is tied to the choice of  $\mathcal{N} = 4$  out of  $\mathcal{N} = 8$ . On the other hand, this reflects the choice of the complex structure on the hyperKähler vacuum manifold, and should not matter for the shape of the Hodge diamond.

For an illustration, consider ADHM quantum mechanics with  $U(k)$  gauge group and  $U(N)$  flavor group, i.e., the D0-D4 system with  $k$  D0's and  $N$  D4's. For these theories, FI constants are allowed and lift the would-be Coulombic vacua, so the gapless asymptotic flat direction arises entirely from hypermultiplets only, and the twisted partition function can compute a proper integral index. Furthermore,  $\mathcal{N} = 8$  implies that the FI constants can be thought of as a triplet, so that  $\zeta = 0$  does not divide FI constant space into two domains. No wall-crossing can occur, as is also evident from the explicit computations. The classical moduli space will be of the form  $M_{k;N} \times \mathbb{R}^4$  where the latter is the decoupled trace part of the adjoint hypermultiplet while the former is the moduli space of centered  $k$  instantons of  $U(N)$  gauge theory. Will the same procedure as in the previous examples compute  $L^2$  Index of  $M_{k;N}$ ?

Theories for a single D0 are the simplest. For example,  $k = 1$  and  $N = 1$  gives

$$\Omega_{\text{ADHM}}^{k=1;N=1}(\mathbf{y}, \mathbf{z}, \tilde{x}) = \frac{(\mathbf{y}\mathbf{z}^{-1}\tilde{x}^{1/2} - \mathbf{y}^{-1}\mathbf{z}\tilde{x}^{-1/2})(\mathbf{y}^{-1}\mathbf{z}\tilde{x}^{1/2} - \mathbf{y}\mathbf{z}^{-1}\tilde{x}^{-1/2})}{(\mathbf{z}\tilde{x}^{1/2} - \mathbf{z}^{-1}\tilde{x}^{-1/2})(\mathbf{z}^{-1}\tilde{x}^{1/2} - \mathbf{z}\tilde{x}^{-1/2})}, \quad (3.22)$$

where  $\tilde{x}$  is from the flavor  $U(1)$  of the adjoint hypermultiplet. This happens to equal

that of a single free hypermultiplet,<sup>7</sup> which in the past encouraged a single bound state interpretation. However, the matter is a bit more subtle, as we presently explore. Taking flavor neutral sector by expanding in either  $\tilde{x}$  or  $1/\tilde{x}$  and setting either to zero gives  $\tilde{\Omega}(\mathbf{y}, \mathbf{z}) = 1$ , which we are encouraged to interpret as

$$\mathcal{I}_{\text{ADHM}}^{k=1;N=1} \Big|_{L^2} = 1 . \quad (3.25)$$

More generally we find

$$\begin{aligned} \tilde{\Omega}_{\text{ADHM}}^{k=1;N}(\mathbf{y}, \mathbf{z} \rightarrow 0) &= 1 + \mathbf{y}^2 + \dots \mathbf{y}^{2N-2} , \\ \tilde{\Omega}_{\text{ADHM}}^{k=1;N}(\mathbf{y}, \mathbf{z} \rightarrow \infty) &= 1 + \mathbf{y}^{-2} + \dots \mathbf{y}^{2-2N} , \end{aligned} \quad (3.26)$$

which contain a single common sector corresponding to  $1 \cdot \mathbf{y}^0$  but otherwise, again, disagree with each other.

Recall that the ADHM moduli space has the form  $M_{\text{ADHM}}^{k=1;N} \times \mathbb{R}^4$ , where the latter factor is the free center of mass part and  $M_{\text{ADHM}}^{k=1;N}$  is  $4(N-1)$  dimensional manifold that is contractible to  $\mathbb{P}^{N-1}$ . Thus, we have

$$H^\bullet(M_{\text{ADHM}}^{k=1;N}) = H^\bullet(\mathbb{P}^{N-1}) ,$$

while

$$H^\bullet(M_{\text{ADHM}}^{k=1;N}, \partial M_{\text{ADHM}}^{k=1;N})$$

is the Poincare dual thereof. Thus,  $L^2$  cohomology of  $M_{\text{ADHM}}^{k=1;N}$ , again using eq. (3.10), is

$$H_{L^2}^{n=2N-2}(M_{\text{ADHM}}^{k=1;N}) = \mathbb{R} , \quad (3.27)$$

and vanishes for other  $n$ 's, such that

$$\mathcal{I}_{\text{ADHM}}^{k=1;N} \Big|_{L^2} = 1 . \quad (3.28)$$

We again find that the common sector of the two expansions of  $\tilde{\Omega}_{\text{ADHM}}^{k=1;N}(\mathbf{y}, \mathbf{z})$  captures precisely the same answer as this.

Note that this result is qualitatively on par with  $\mathcal{N} = 4$  examples and show no real advantage due to larger supersymmetry. There is some simplification in that the two expansions with respect to  $\tilde{x}$  and  $1/\tilde{x}$  coincide, where  $\tilde{x}$  is associated with the flavor symmetry

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<sup>7</sup>Expansions in  $1/\tilde{x}$  and in  $\tilde{x}$  produce

$$1 + [(\mathbf{z}^{-1} - \mathbf{z})^2 - (\mathbf{z}/\mathbf{y} - \mathbf{y}/\mathbf{z})^2] \tilde{x}^{-1} + O(\tilde{x}^{-2}) , \quad (3.23)$$

and

$$1 + [(\mathbf{z}^{-1} - \mathbf{z})^2 - (\mathbf{z}/\mathbf{y} - \mathbf{y}/\mathbf{z})^2] \tilde{x} + O(\tilde{x}^2) , \quad (3.24)$$

so that, unlike the case of a free chiral multiplet, the presence of a free hypermultiplet gives 1 in the end in our expansions relative to flavor and  $\tilde{R}$  chemical potential. In the context of ADHM, the trace part of the adjoint hypermultiplet decouples and represents the center of mass motion of the instantons, yet, as far as this procedure goes, is pretty harmless.

in  $\mathcal{N} = 8$  sense. However, the expansions for  $\mathbf{z}$  and  $1/\mathbf{z}$  do not agree with each other as in  $\mathcal{N} = 4$  examples, and true Index emerges only after taking the common sector appearing in both expansions. This tells us that  $\mathcal{N} = 8$  does not buy us too much; existing claims about bound states based on twisted partition function with chemical potential must be thus taken with a large grain of salt.

### 3.4 General ADHM and multi-particle contributions

Turning to  $k > 1$ , however, we see that such a pattern does not persist. Repeating the same procedure, and considering  $N = 1$  only, we find

$$\tilde{\Omega}_{\text{ADHM}}^{k;N=1}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) = 1, 2, 3, 5, 7, \dots \quad \text{for } k = 1, 2, 3, 4, 5, \dots \quad (3.29)$$

With this new example, we see immediately that the previous instances of various flavor expansions of  $\Omega$  or  $\tilde{\Omega}$  giving some version of cohomology were accidental. The moduli space  $M_{\text{ADHM}}^{k;N=1}$  is  $4k - 4$  real-dimensional, and its cohomology must at least have  $H_0(M)$  and  $H^{4k-4}(M, \partial M)$  nontrivial. Neither of such cohomologies is reflected in the above expansions.

One could be in principle encouraged by the mere integers unambiguously showing up in eq. (3.29), but the numbers are in a clear conflict with M-theory predictions of unique  $L^2$  ground state. In fact, the above numbers in  $\tilde{\Omega}_{\text{ADHM}}^{k;N=1}$  are nothing but the number of partitions for  $k$ , suggesting that each asymptotic sector with bound states of  $k' < k$  D0 branes also contribute 1 each. One way to phrase this suggestion starts with the generating function [37],

$$\tilde{G}_{\text{ADHM}}^{(N=1)}(q; \mathbf{y}, \mathbf{z}) = \sum_{k=1}^{\infty} q^k \tilde{\Omega}_{\text{ADHM}}^{k;N=1}(\mathbf{y}, \mathbf{z}), \quad (3.30)$$

of which Plethystic logarithm gives

$$G_{\text{ADHM}}^{(N=1)}(q; \mathbf{y}, \mathbf{z}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \left[ \tilde{G}_{\text{ADHM}}^{(N=1)}(q^n; \mathbf{y}^n, \mathbf{z}^n) \right], \quad (3.31)$$

where the Möbius function  $\mu$  is defined as

$$\mu(n) = \begin{cases} (-1)^p & \text{if } n \text{ is a product of } p \geq 0 \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

The proposal boils down to the statement that, expanding  $G_{\text{ADHM}}^{(N=1)}$  again in terms of  $q$ ,

$$G_{\text{ADHM}}^{(N=1)}(q; \mathbf{y}, \mathbf{z}) = \sum_{k=1}^{\infty} q^k \Lambda_{\text{ADHM}}^{k;N=1}(\mathbf{y}, \mathbf{z}), \quad (3.32)$$

gives us a new set of index-like quantities  $\Lambda_{\text{ADHM}}^{k;N=1}(\mathbf{y}, \mathbf{z})$  that carries the information of one-particle bound state. Since  $\tilde{\Omega}_{\text{ADHM}}^{k;N=1}$  happen to be mere numbers, the same is true of  $\Lambda_{\text{ADHM}}^{k;N=1}$ , and one finds

$$\Lambda_{\text{ADHM}}^{k;N=1}(\mathbf{y}, \mathbf{z}) = 1 \quad \text{for all } k \geq 1, \quad (3.33)$$

which is taken to imply

$$\mathcal{I}_{\text{ADHM}}^{k;N=1} \Big|_{L^2} = 1 \quad \text{for all } k \geq 1, \quad (3.34)$$

as anticipated from M-theory. Of course this does not really explain why each partition of  $k$  contributes exactly unit; it merely offers an observation of a natural relationship between  $\mathcal{I}_{\text{ADHM}}^{k;N=1}|_{L^2}$ 's and  $\Omega_{\text{ADHM}}^{(k' \leq k; N=1)}$ 's. It also tells us that extraction of  $\mathcal{I}_{L^2}$  index from  $\Omega$ , if any such universal routine does exist, could be a bit more involved than what we have seen in the earlier Abelian examples.

Furthermore, ADHM for  $U(N)$  instantons with  $N > 1$  offers even more challenges, as the end results cannot be encoded in a naive generalization of the  $N = 1$  sequence. Let us list some of the results for  $\tilde{\Omega}_{\text{ADHM}}^{k;N}$ :

$$\begin{aligned} \tilde{\Omega}_{\text{ADHM}}^{k=1;N=2}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 1 + \mathbf{y}^{\pm 2}, \\ \tilde{\Omega}_{\text{ADHM}}^{k=2;N=2}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 3 + 2\mathbf{y}^{\pm 2}, \\ \tilde{\Omega}_{\text{ADHM}}^{k=3;N=2}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 5 + 5\mathbf{y}^{\pm 2}, \\ \tilde{\Omega}_{\text{ADHM}}^{k=4;N=2}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 10 + 9\mathbf{y}^{\pm 2} + \mathbf{y}^{\pm 4}, \end{aligned} \quad (3.35)$$

$$\begin{aligned} \tilde{\Omega}_{\text{ADHM}}^{k=1;N=3}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 1 + \mathbf{y}^{\pm 2} + \mathbf{y}^{\pm 4}, \\ \tilde{\Omega}_{\text{ADHM}}^{k=2;N=3}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 3 + 3\mathbf{y}^{\pm 2} + \mathbf{y}^{\pm 4}, \\ \tilde{\Omega}_{\text{ADHM}}^{k=3;N=3}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 6 + 7\mathbf{y}^{\pm 2} + 7\mathbf{y}^{\pm 4} + 2\mathbf{y}^{\pm 6}, \\ \tilde{\Omega}_{\text{ADHM}}^{k=1;N=4}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) &= 1 + \mathbf{y}^{\pm 2} + \mathbf{y}^{\pm 4} + \mathbf{y}^{\pm 6}, \end{aligned} \quad (3.36)$$

$$\tilde{\Omega}_{\text{ADHM}}^{k=2;N=4}(\mathbf{y}, \mathbf{z}^{\pm 1} \rightarrow 0) = 3 + 3\mathbf{y}^{\pm 2} + 4\mathbf{y}^{\pm 4} + 3\mathbf{y}^{\pm 6} + \mathbf{y}^{\pm 8}. \quad (3.37)$$

The common flavor neutral parts are represented by single integers, in each case of  $(k, N)$  ADHM, which are 1, 3, 5, 10, ... for  $N = 2$ ; 1, 3, 6, ... for  $N = 3$ ; 1, 3, ... for  $N = 4$ . The M-theory prediction is again a unique  $L^2$  bound state for all of these ADHM theories, so for  $k > 1$ , multi-particle interpretation is again needed for this to make sense, although, unlike  $N = 1$  example, simple enumeration of partitions of  $k$  does not suffice.

To summarize, we have seen that physical  $L^2$  boundary condition cannot be mimicked by introduction of chemical potentials as infrared regulators. In some cases, the latter twisted partition functions capture some version of cohomologies, depending on which Taylor expansion is taken, whose common flavor singlet parts turned out to equal the desired  $L^2$  spectra. However, as we have seen in ADHM examples, this particular pattern proves to be accidental; even though example by example  $L^2$  spectrum seems to be embedded in  $\Omega$  in one way or another, no clear and general dictionary between  $\Omega$  and  $\mathcal{I}$  appears

to exist. The situation does not improve upon imposing larger  $\mathcal{N} = 8$  supersymmetry, and one finds again ambiguous flavor-singlet contents typically. Nevertheless, the mere fact that  $L^2$  spectrum is embedded in  $\Omega$  in some indirect manner is by itself pretty miraculous, given how callously the chemical potential deals with the subtle infrared issues. Relationships between  $\Omega$  and  $\mathcal{I}$  we have uncovered might hint at a more universal relationship for noncompact dynamics. Further investigation along this line is desired.

## 4 Bulk, boundary, and pure Yang-Mills

So far we have explored examples with gapless asymptotic directions coming from matter multiplets. In these examples, chemical potentials can lift all such directions, resulting in integral  $\Omega$ 's, which nevertheless must be interpreted with much care. When gapless directions come from vector multiplets, however, there arise further issues. For instance, if no FI constants can be turned on, there would be rank-many Coulombic flat directions that cannot be lifted by chemical potential. With such theories, suppose that we still take the final expressions of type (2.3) at face value. What have we computed?

Note that the formula (2.3) is independent of  $\beta$ . At least naively, this is hardly surprising, as Index counts only  $H = 0$  states. However, the truth of the matter is more complicated. For theories with asymptotically flat directions, gapped or gapless, the continuum sector with  $E > E_{\text{gap}}$  can contribute to the path integral in such a way that the twisted partition function acquires  $\beta$ -dependence. As we noted already, we first try to deform the theory such that  $\beta E_{\text{gap}} \rightarrow \infty$  to remove such a continuum contribution whenever possible. On the other hand, this is not possible for gapless asymptotic directions, and we may acquire contributions from continuum that touches  $E = 0$  and  $\Omega$  would be  $\beta$ -dependent. Such directions from the chiral sector can be controlled by introducing flavor chemical potentials, which lead to both problems and some promises as described already. For such directions from a vector multiplet, however, even this is not possible. We must thus expect  $\beta$ -dependence of  $\Omega$  in general, and the formula (2.3), being independent of  $\beta$ , must be a certain limit of  $\Omega(\beta)$  for such gapless theories.

It would have been nice if the formula (2.3) had computed the  $\beta \rightarrow \infty$  limit of  $\Omega$ , with respect to some boundary condition. Since we do encounter non-integral  $\Omega$ , however, this is not possible; true Index, regardless of boundary condition, would have to be integral. The obvious answer is then that we have computed the other limit of the twisted partition function at  $\beta \rightarrow 0$ , or equivalently the “bulk” part of the Index. For pure Yang-Mills theory, this is easiest to see: since the theory is not compact, the only other parameter that enters the computation is the electric coupling  $e^2$  with the dimension of mass cubed. The only dimensionless combination,  $e^{2/3}\beta$ , vanishes in the localization limit of  $e^2 \rightarrow 0$ , implying that the localization formula (2.3) effectively computes the bulk part of the Index,

$$\mathcal{I}_{\text{bulk}} = \Omega \Big|_{e^{2/3}\beta \rightarrow 0}.$$

For simplicity, we will continue to use the notation  $\Omega$ , for the twisted partition function computed by eq. (2.3).

The bulk part of the Index is often non-integral. Contribution from the continuum sector must be computed separately, say,  $-\delta\mathcal{I}$ , and subtracted from this to produce the true Index,

$$\mathcal{I} = \mathcal{I}_{\text{bulk}} + \delta\mathcal{I} .$$

Is there a systematic way to compute  $\delta\mathcal{I}$ ? Since  $L^2$  is natural for path integral, the boundary contribution with the  $L^2$  condition might carry its own physical meaning. In the well-known D0 bound state problem [38–40] decades ago, exactly such an interpretation was found [39], which lead to a method of computing the boundary contribution of a theory as the bulk contribution of a different, much simpler theory. We will see later that this behavior is quite prevalent, and in particular quite generic in the context of BPS quivers and the associated wall-crossing phenomena.

In this section, we will first concentrate on a simplest possible class of systems, the  $\mathcal{N} = 4, 8, 16$  pure Yang-Mills quantum mechanics, where the bulk and the boundary are related.

#### 4.1 $\mathcal{N} = 4, 8$ pure Yang-Mills

For pure Yang-Mills with a simple gauge group, the Fayet-Iliopoulos constant is absent, whereby we lose the trick of  $Q_\infty = -\zeta$ . Nevertheless, one can still argue that the residue contribution from poles at infinities of  $(\mathbb{C}^*)^r$  cancel among themselves.<sup>8</sup> The localization procedure gives, then,

$$\Omega^G(\mathbf{y}, x; e^{2/3}\beta \rightarrow 0) = \frac{1}{|W|} \text{JK-Res}_\eta^{\text{internal}} g_G(u) \, d^r u ,$$

where  $G$  labels the gauge group. Contributing singularities would in general include highly degenerate ones, for which we resorted to the constructive procedure outlined in appendix A.1.

It turns out that, for pure  $\mathcal{N} = 4$   $G$ -gauged quantum mechanics, this bulk contribution to the index, or the  $e^{2/3}\beta \rightarrow 0$  limit of twisted partition function can be organized into a universal formula,

$$\Omega_{\mathcal{N}=4}^G(\mathbf{y}) = \frac{1}{|W|} \sum_w' \frac{1}{\text{Det}(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} , \quad (4.1)$$

where the sum is only over the elliptic Weyl elements and  $|W|$  is the cardinality of the Weyl group itself. An elliptic Weyl element  $w$  is defined by absence of eigenvalue 1. In other words, in the canonical  $r$ -dimensional representation of the Weyl group on the weight lattice,

$$\text{Det}(1 - w) \neq 0 .$$

For pure  $\mathcal{N} = 8$   $G$ -gauged quantum mechanics, obtained by adding to the  $\mathcal{N} = 4$  theory an adjoint chiral, we have a flavor chemical potential  $x$  of the adjoint to play with as well. With  $R = 0$  for the adjoint chiral, we have

$$\Omega_{\mathcal{N}=8}^G(\mathbf{y}, x) = \frac{1}{|W|} \sum_w' \frac{1}{\text{Det}(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \frac{\text{Det}(\mathbf{y}^{-1}x^{1/2} - \mathbf{y}x^{-1/2} \cdot w)}{\text{Det}(x^{1/2} - x^{-1/2} \cdot w)} , \quad (4.2)$$

where again the sum is over the elliptic Weyl elements of  $G$ .

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<sup>8</sup>See appendix A.2.

Why such a universal and simple formula? This can be motivated by the statement that  $\mathcal{N} = 4, 8$  pure Yang-Mills quantum mechanics has no bound state, i.e., the true  $L^2$  index is zero. We know that this is the case at least for  $G = \text{SU}(N)$ . D2/D3-branes multiply-wrapped on  $S^2$  and  $S^3$  in noncompact Calabi-Yau two-fold and three-fold are governed by such dynamics. Judging from how such geometrically engineered  $d = 4$   $\mathcal{N} = 2, 4$  Yang-Mills field theory behaves, we can pretty much rule out physical bound states of many identical D-branes wrapping on  $S^2$  and  $S^3$ . The worldvolume dynamics on such multiply-wrapped D-brane are  $\mathcal{N} = 4, 8$  pure  $\text{SU}(N)$  quantum mechanics, which tells us the Witten index of these theories must vanish. If we assume that the latter statement extends to all gauge groups, we have

$$\mathcal{I}_{\mathcal{N}=4,8}^G = \Omega_{\mathcal{N}=4,8}^G(e^{2/3}\beta \rightarrow 0) + \delta\mathcal{I}_{\mathcal{N}=4,8}^G = 0, \quad (4.3)$$

where  $\delta\mathcal{I}$  is the contribution associated with the continuum sector. This implies

$$\Omega_{\mathcal{N}=4,8}^G(e^{2/3}\beta \rightarrow 0) = \mathcal{I}_{\text{bulk}} = -\delta\mathcal{I}_{\mathcal{N}=4,8}^G. \quad (4.4)$$

We have transformed, apparently, a cumbersome problem of computing the bulk contribution to  $\mathcal{I}$  into an even more difficult problem of computing continuum contribution  $\delta\mathcal{I}$ . Why is this helpful?

This simple observation can be made into a computational tool as follows [39], based on the observation that  $\delta\mathcal{I}$  is entirely a boundary contribution. Since  $\delta\mathcal{I}$  depends only on the dynamics at the asymptotic region, we need to ask how the latter looks like. The low energy theory in the Coulomb phase is free with the target being the asymptotic part of the orbifold,

$$\mathcal{O}(G) = \mathbb{R}^{3r}/W \text{ or } \mathbb{R}^{5r}/W, \quad (4.5)$$

for  $\mathcal{N} = 4, 8$ , respectively, where  $W$  is the Weyl group of  $G$ . Let us then consider  $\mathcal{N} = 4, 8$   $\text{U}(1)^r$  gauge theory without matter, and gauged by the discrete group  $W$  acting on the  $\text{U}(1)$ 's. The latter is a valid quantum theory of its own, and shares the same asymptotic dynamics as the original Yang-Mill theory. We will denote this free theory also by the same symbol  $\mathcal{O}(G)$ . Since the interacting  $G$ -Yang-Mills theory and the free  $\mathcal{O}(G)$  Abelian theory shares the same asymptotic dynamics, we conclude that

$$\delta\mathcal{I}^G = \delta\mathcal{I}^{\mathcal{O}(G)}. \quad (4.6)$$

On the other hand, the free orbifold  $\mathcal{O}(G)$  cannot possibly have a bound state either when  $G$  theory does not, since the former can be considered a limit of the latter, so its bulk contribution to the Witten index is such that

$$\mathcal{I}_{\text{bulk}}^{\mathcal{O}(G)} = -\delta\mathcal{I}^{\mathcal{O}(G)}, \quad (4.7)$$

which finally brings us to

$$\Omega_{\mathcal{N}=4,8}^G(e^{2/3}\beta \rightarrow 0) = -\delta\mathcal{I}_{\mathcal{N}=4,8}^G = -\delta\mathcal{I}_{\mathcal{N}=4,8}^{\mathcal{O}(G)} = \left(\mathcal{I}_{\mathcal{N}=4,8}^{\mathcal{O}(G)}\right)_{\text{bulk}}. \quad (4.8)$$



The right-most expression can be easily evaluated using the Heat Kernel regularization, when  $\mathbf{y} = 1$  and  $x = 1$ , following the  $SU(2)$  case in ref. [39], and this gave [41, 42]

$$\frac{1}{|W|} \sum_w' \frac{1}{\text{Det}(1 - w)}, \quad (4.9)$$

for all  $\mathcal{N}$ .<sup>9</sup> The expressions in eqs. (4.1) and (4.2) for  $\mathcal{N} = 4, 8$  pure Yang-Mills dynamics are merely equivariant generalizations of this result.  $G = SU(N)$  cases already have ample and independent evidences that argue against threshold bound states, and the above line of thinking, combined with our direct localization computation, further supports the same conclusion. With this class of examples understood, the mere fact that we recover exactly the same type of rational structure for other  $G$ 's involving only the elliptic Weyl elements, from the brute-force localization computation, indicates convincingly that threshold bound states are absent for general  $G$  as well.

We will later see, when we turn to  $\mathcal{N} = 16$  and also to  $\mathcal{N} = 4$  nonprimitive quivers, how this rational structure expands in a very logical manner in the presence of threshold bound states. For example, for  $\mathcal{N} = 16$  where threshold bound states are generally expected,  $\delta\mathcal{I}_{\mathcal{N}=16}^G$  can be seen to receive contributions from continuum sectors involving partial bound states. Thus, one ends up with a recursive form of  $\mathcal{I}_{\text{bulk}}^G = \Omega_{\mathcal{N}=16}^G(e^{2/3}\beta \rightarrow 0)$ , built up from  $\mathcal{I}_{\mathcal{N}=16}^{\tilde{G}}$  and  $\mathcal{I}_{\text{bulk}}^{\mathcal{O}(\tilde{G})}$  where  $\tilde{G}$ 's are subgroups of  $G$ . We will come back to this story in section 4.2.

#### 4.1.1 Examples

We have checked the results above against computations following HKY derivation of  $\Omega$  [15]. Details of the numerical computation is often too lengthy to describe, as it can easily involve thousands of flags and singularities with contributing residues; we merely state here that *the predictions in eq. (4.1) and in eq. (4.2) have been confirmed*, by such explicit evaluations of the relevant JK residue formulae, up to rank 5 for  $\mathcal{N} = 4$  theories and up to rank 4 for  $\mathcal{N} = 8$  except  $F_4$ . This gives us enough confidence to believe the heuristic arguments presented above hold true. Here we display the resulting  $\Omega$ 's explicitly.

**$\mathcal{N} = 4$ .** For  $SU(N)$  with the permutation group  $S_N$  as the Weyl group, the only elliptic Weyl elements are the fully cyclic ones, such as  $(123 \cdots N)$  that permutes  $1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow N \rightarrow 1$ . There are  $(N-1)!$  number of such an  $N$ -cyclic permutations, belonging to a single conjugacy class, each of which contributes

$$\frac{1}{\text{Det}(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} = \frac{1}{\mathbf{y}^{-N+1} + \mathbf{y}^{-N+3} + \cdots + \mathbf{y}^{N-1}},$$

where one evaluated the left hand side on the irreducible  $(N-1)$  dimensional representation. This gives

- $SU(N)$

$$\Omega_{\mathcal{N}=4}^{SU(N)}(\mathbf{y}) = \frac{1}{|W|} \sum_w' \frac{1}{\text{Det}(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} = \frac{1}{N} \frac{\mathbf{y}^{-1} - \mathbf{y}}{\mathbf{y}^{-N} - \mathbf{y}^N}. \quad (4.10)$$

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<sup>9</sup> $\mathcal{N} = 2$  case also gives the same expression for  $\mathcal{I}_{\text{bulk}}^{\mathcal{O}(G)}$  but its relation to  $\delta\mathcal{I}_{\mathcal{N}=2}^G$  is no longer justified due to the logarithmic nature of the wavefunctions.

For more general gauge groups, however, there will be more than one such conjugacy classes contributing.

For  $\text{SO}(2N+1)$  and  $\text{Sp}(N)$  and for  $\text{SO}(2N)$ , the Weyl groups are  $S_Z \ltimes (Z_2)^N$  and  $S_N \ltimes (Z_2)^{N-1}$ , respectively. The elliptic Weyl elements have the general form,

$$(\dot{a}\dot{b}\dot{c}d\dots)(kl\dot{m}n\dots)\dots$$

where  $(\dots)$  again represents a cyclic permutation and each dot above a label means a sign flip. The number of such dots in each cyclic factor must be all odd. For  $\text{SO}(2N)$ , the total number of dots is even. The relevant determinant factorizes in terms of those of such cyclic factors,

$$\frac{1}{\text{Det}(y^{-1} - y \cdot w)} \Big|_{w=(1,\dot{2},\dots,\dot{k},k+1,\dots,n)} = \frac{1}{\mathbf{y}^{-n} + (-1)^{k+1}\mathbf{y}^n},$$

which also shows why we need odd  $k$  for  $w$  to be elliptic. Once the  $n$  labels are determined and  $k$  fixed, we can count elements within such a class. Permutation among the chosen  $n$  labels gives  $(n-1)!$  possible maximally cyclic permutation among these  $n$  labels, and for each choice, assignment of dots gives a further  ${}_nC_k$  choices. Unlike  $\text{SU}(N)$ , there seems to be no simple and closed form for general ranks, so here we list the answer for ranks up to five.

- $\text{SO}(4)$

$$\Omega_{\mathcal{N}=4}^{\text{SO}(4)}(\mathbf{y}) = \frac{1}{4} \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} \quad (4.11)$$

- $\text{SO}(5)$  and  $\text{Sp}(2)$

$$\Omega_{\mathcal{N}=4}^{\text{SO}(4)}(\mathbf{y}) = \Omega_{\mathcal{N}=4}^{\text{Sp}(2)}(\mathbf{y}) = \frac{1}{8} \left[ \frac{2}{\mathbf{y}^{-2} + \mathbf{y}^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} \right] \quad (4.12)$$

- $\text{SO}(6)$

$$\Omega_{\mathcal{N}=4}^{\text{SO}(6)}(\mathbf{y}) = \frac{1}{24} \frac{6}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})} \quad (4.13)$$

- $\text{SO}(7)$  and  $\text{Sp}(3)$

$$\begin{aligned} \Omega_{\mathcal{N}=4}^{\text{SO}(7)}(\mathbf{y}) &= \Omega_{\mathcal{N}=4}^{\text{Sp}(3)}(\mathbf{y}) = \\ &= \frac{1}{48} \left[ \frac{8}{\mathbf{y}^{-3} + \mathbf{y}^3} + \frac{6}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^3} \right] \end{aligned} \quad (4.14)$$

- $\text{SO}(8)$

$$\begin{aligned} \Omega_{\mathcal{N}=4}^{\text{SO}(8)}(\mathbf{y}) &= \\ &= \frac{1}{192} \left[ \frac{32}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{12}{(\mathbf{y}^{-2} + \mathbf{y}^2)^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^4} \right] \end{aligned} \quad (4.15)$$

- $\text{SO}(9)$  and  $\text{Sp}(4)$

$$\begin{aligned}\Omega_{\mathcal{N}=4}^{\text{SO}(9)}(\mathbf{y}) &= \Omega_{\mathcal{N}=4}^{\text{Sp}(4)}(\mathbf{y}) = \\ &= \frac{1}{384} \left[ \frac{48}{\mathbf{y}^{-4} + \mathbf{y}^4} + \frac{32}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-1} + \mathbf{y})} \right. \\ &\quad \left. + \frac{12}{(\mathbf{y}^{-2} + \mathbf{y}^2)^2} + \frac{12}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^4} \right]\end{aligned}\quad (4.16)$$

- $\text{SO}(10)$

$$\begin{aligned}\Omega_{\mathcal{N}=4}^{\text{SO}(10)}(\mathbf{y}) &= \\ &= \frac{1}{1920} \left[ \frac{240}{(\mathbf{y}^{-4} + \mathbf{y}^4)(\mathbf{y}^{-1} + \mathbf{y})} \right. \\ &\quad \left. + \frac{160}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-2} + \mathbf{y}^2)} + \frac{20}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})^3} \right]\end{aligned}\quad (4.17)$$

- $\text{SO}(11)$  and  $\text{Sp}(5)$

$$\begin{aligned}\Omega_{\mathcal{N}=4}^{\text{SO}(11)}(\mathbf{y}) &= \Omega_{\mathcal{N}=4}^{\text{Sp}(5)}(\mathbf{y}) = \\ &= \frac{1}{3840} \left[ \frac{384}{\mathbf{y}^{-5} + \mathbf{y}^5} + \frac{240}{(\mathbf{y}^{-4} + \mathbf{y}^4)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{160}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-2} + \mathbf{y}^2)} \right. \\ &\quad + \frac{80}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-1} + \mathbf{y})^2} + \frac{60}{(\mathbf{y}^{-2} + \mathbf{y}^2)^2(\mathbf{y}^{-1} + \mathbf{y})} \\ &\quad \left. + \frac{20}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})^3} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^5} \right]\end{aligned}\quad (4.18)$$

For exceptional groups, the Weyl groups and  $\mathcal{I}_{\text{bulk}}^{\mathcal{O}(G)}$  are more complicated. Here we display the results for  $G_2$  and  $F_4$ , whose Weyl group and elliptic Weyl elements are also summarized in appendix B.

- $G_2$

$$\Omega_{\mathcal{N}=4}^{G_2}(\mathbf{y}) = \frac{1}{12} \left[ \frac{2}{\mathbf{y}^{-2} - 1 + \mathbf{y}^2} + \frac{2}{\mathbf{y}^{-2} + 1 + \mathbf{y}^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} \right]\quad (4.19)$$

- $F_4$

$$\begin{aligned}\Omega_{\mathcal{N}=4}^{F_4}(\mathbf{y}) &= \\ &= \frac{1}{1152} \left[ \frac{144}{\mathbf{y}^{-4} + \mathbf{y}^4} + \frac{96}{\mathbf{y}^{-4} - 1 + \mathbf{y}^4} + \frac{64}{\mathbf{y}^{-4} + \mathbf{y}^{-2} + \mathbf{y}^2 + \mathbf{y}^4} \right. \\ &\quad + \frac{12}{(\mathbf{y}^{-2} + \mathbf{y}^2)^2} + \frac{16}{(\mathbf{y}^{-2} - 1 + \mathbf{y}^2)^2} + \frac{16}{(\mathbf{y}^{-2} + 1 + \mathbf{y}^2)^2} \\ &\quad \left. + \frac{36}{(\mathbf{y}^{-1} + \mathbf{y})^2(\mathbf{y}^{-2} + \mathbf{y}^2)} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^4} \right]\end{aligned}\quad (4.20)$$

$\mathcal{N} = 8$ . Their analogues for  $\mathcal{N} = 8$  theories are equally straightforward. Given  $\Omega_{\mathcal{N}=4}^G$  expressed as a sum over the conjugacy classes,  $\mathcal{C}$ , of elliptic Weyl element in  $W(G)$ ,

$$\Omega_{\mathcal{N}=4}^G = \sum_{\mathcal{C}}' \frac{A_{\mathcal{C}}}{P_{\mathcal{C}}(\mathbf{y})},$$

with rational numbers  $A_{\mathcal{C}}$  and the symmetric Laurent polynomials  $P_{\mathcal{C}}$ , of degree  $r$ , its  $\mathcal{N} = 8$  counterpart is given as

$$\Omega_{\mathcal{N}=8}^G = \sum_{\mathcal{C}}' \frac{A_{\mathcal{C}}}{P_{\mathcal{C}}(\mathbf{y})} \cdot \frac{P_{\mathcal{C}}(x^{-1/2}\mathbf{y})}{P_{\mathcal{C}}(x^{-1/2})}, \quad (4.21)$$

with the unit flavor charge and zero  $R$ -charge for the adjoint chiral. For instance,  $\text{SO}(4)$  case works as

$$\Omega_{\mathcal{N}=4}^{\text{SO}(4)} = \frac{1/4}{(\mathbf{y}^{-1} + \mathbf{y})^2} \rightarrow \Omega_{\mathcal{N}=8}^{\text{SO}(4)} = \frac{1/4}{(\mathbf{y}^{-1} + \mathbf{y})^2} \cdot \frac{(x^{1/2}\mathbf{y}^{-1} + x^{-1/2}\mathbf{y})^2}{(x^{1/2} + x^{-1/2})^2}. \quad (4.22)$$

## 4.2 $\mathcal{N} = 16$ pure Yang-Mills

With the maximal supersymmetry of  $\mathcal{N} = 16$ , threshold bound states are in general expected, as  $\text{SU}(N)$  case corresponds to the well-known multi-D0 bound state problem [38]. Therefore the twisted partition function must include integral contribution from such  $L^2$  states. This also means that the continuum sectors can be more involved for higher rank theories, since more than one partial bound states may form and new continuum contributions from such sectors can enter  $\Omega = \mathcal{I}_{\text{bulk}}$ .

For  $\text{SU}(N)$ , at the numerical level,  $\mathcal{I}_{\text{bulk}}$  and its general structure has been identified long time ago, which motivated this study to begin with. The numerical limit of the twisted partition function was known to be [4]

$$\Omega_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1; x \rightarrow 1} = \sum_{p|N} \frac{1}{p^2}. \quad (4.23)$$

We have rederived this expression by explicit computation of fully equivariant version for some low-rank examples. With three adjoint chirals  $\Phi_a$ 's, we can assign  $R$ -charges  $(R_1, R_2, R_3)$  and turn on one  $\text{U}(1)$  flavor chemical potential with charges  $(F_1, F_2, F_3)$ , so that the leading superpotential is trilinear  $\sim \Phi_1 \Phi_2 \Phi_3$ , and find

$$\Omega_{\mathcal{N}=16}^{\text{SU}(N)} = 1 + \sum_{p|N; p>1} \Delta_{\mathcal{N}=16}^{\text{SU}(p)}, \quad (4.24)$$

where  $\Delta_{\mathcal{N}=16}^G$  are the direct analog of the expressions for  $\Omega_{\mathcal{N}=8}^G$  on the right hand side of eq. (4.21). They are sums over conjugacy classes,  $\mathcal{C}$ , of elliptic Weyl elements of  $G$ ,

$$\Delta_{\mathcal{N}=16}^G \equiv \sum_{\mathcal{C}}' \frac{A_{\mathcal{C}}}{P_{\mathcal{C}}(\mathbf{y})} \cdot \prod_{a=1}^3 \frac{P_{\mathcal{C}}(x^{F_a/2}\mathbf{y}^{(R_a-2)/2})}{P_{\mathcal{C}}(x^{F_a/2}\mathbf{y}^{R_a/2})}. \quad (4.25)$$

Note that the limit  $x \rightarrow 1$  is well-defined. This happy outcome, however, may be attributed to the fact that we are cheating a little here for  $\mathcal{N} = 16$ . The trilinear superpotential  $\sim \Phi_1 \Phi_2 \Phi_3$  does not guarantee the commutator superpotential  $\text{tr}(\Phi_1[\Phi_2, \Phi_3])$ . Thus, in principle, we are computing  $\Omega$ 's for a deformed version of  $\mathcal{N} = 16$  pure Yang-Mills theory. Nevertheless, we believe this suffices for the purpose of Witten Index computation.

For one thing, the physical origin of eq. (4.23), now elevated to its equivariant version (4.24), has been understood in very physical terms:  $p = 1$  gives 1, corresponding to the genuine  $L^2$  index of this quantum mechanics, while the others with  $p > 1$  come from asymptotic dynamics of a sector where the  $p$  partial bound states associated with  $p$  number of  $\text{SU}(N/p)$  subsystems are separated far from one another. In other words,  $\Omega_{\mathcal{N}=16}^{\text{SU}(N)}$  has the simple interpretation in terms of Witten indices as

$$\Omega_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1; x \rightarrow 1} = \mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1} + \sum_{p|N; p>1} \frac{1}{p^2} \times \mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N/p)} \Big|_{\mathbf{y} \rightarrow 1}, \quad (4.26)$$

with  $\mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N)} = 1$  for all  $N$ . The insight that anticipated and explained this formula, originates in ref. [39]. This intriguing structure of the twisted partition function, or the bulk part of the Witten index, persists well beyond pure Yang-Mills theories, as will be explained in the next section. Here it suffices to note that, once we understand these structures of fractional  $\Omega$ 's, true and integral  $L^2$  index can be read off easily from  $\Omega = \mathcal{I}_{\text{bulk}}$ , without having to deal with the very subtle boundary contribution  $\delta\mathcal{I}$ .

We repeated the same exercise for other simple gauge groups also. The fully equivariant  $\Omega_{\mathcal{N}=16}^G$ 's are in eq. (4.27) below, while table 1 lists the numerical limits for  $\mathcal{N} = 4, 8, 16$  for easy comparisons among various theories. In the past, there have been attempts to compute these numbers. After  $\mathcal{N} = 4, 8, 16$   $\text{SU}(2)$  cases were computed [39, 40], Moore, Nekrasov, and Shatashvili (MNS) gave a localization prescription for  $\text{SU}(N)$  [4], with the same result as above. However, later adaptations of MNS for other gauge groups [43, 44] do not agree with the above. Our computation here should be taken to supersede these older results; note that the localization procedure of HKY *derives* the zero mode contour while MNS gave an intelligent guess on the contour. Failure of the various contour “prescriptions” offered in the past is hardly surprising, especially for theories with degenerate singularities for which order of integration is crucial and mutually distinct between different degenerate singularities. Such subtleties manifest also among  $\text{U}(N)$  based theories, with the simplest example we know of being non-Abelian cyclic triangle quivers [45].

The analogues of eq. (4.24), complete with how  $\Omega$  is decomposed in terms of continuum sectors associated with subgroups, are more involved for general gauge groups. For the above low-rank examples, the following decompositions are found,

$$\begin{aligned} \Omega_{\mathcal{N}=16}^{\text{SO}(5)/\text{Sp}(2)} &= 1 + 2\Delta_{\mathcal{N}=16}^{\text{SO}(3)/\text{Sp}(1)} + \Delta_{\mathcal{N}=16}^{\text{SO}(5)/\text{Sp}(2)}, \\ \Omega_{\mathcal{N}=16}^{G_2} &= 2 + 2\Delta_{\mathcal{N}=16}^{\text{SU}(2)} + \Delta_{\mathcal{N}=16}^{G_2}, \\ \Omega_{\mathcal{N}=16}^{\text{SO}(7)} &= 1 + 3\Delta_{\mathcal{N}=16}^{\text{SO}(3)} + \left(\Delta_{\mathcal{N}=16}^{\text{SO}(3)}\right)^2 + \Delta_{\mathcal{N}=16}^{\text{SO}(5)} + \Delta_{\mathcal{N}=16}^{\text{SO}(7)}, \end{aligned}$$

	$\mathcal{N} = 4, 8$	$\mathcal{N} = 16$
$SU(N)$	$\frac{1}{N^2}$	$\sum_{p N} \frac{1}{p^2}$
$SO(4)$	$\frac{1}{16}$	$\frac{25}{16}$
$SO(6) = SU(4)$	$\frac{1}{16}$	$\frac{21}{16}$
$SO(8)$	$\frac{59}{1024}$	$\frac{3755}{1024}$
$SO(5)$	$\frac{5}{32}$	$\frac{53}{32}$
$SO(7)$	$\frac{15}{128}$	$\frac{267}{128}$
$SO(9)$	$\frac{195}{2048}$	$\frac{7555}{2048}$
$Sp(2)$	$\frac{5}{32}$	$\frac{53}{32}$
$Sp(3)$	$\frac{15}{128}$	$\frac{395}{128}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{8067}{2048}$
$G_2$	$\frac{35}{144}$	$\frac{395}{144}$

**Table 1.** The numerical limits of the twisted partition functions,  $\Omega_{\mathcal{N}=4,8,16}^G$  for various gauge groups  $G$ .

$$\begin{aligned}
 \Omega_{\mathcal{N}=16}^{Sp(3)} &= 2 + 3\Delta_{\mathcal{N}=16}^{Sp(1)} + \left(\Delta_{\mathcal{N}=16}^{Sp(1)}\right)^2 + \Delta_{\mathcal{N}=16}^{Sp(2)} + \Delta_{\mathcal{N}=16}^{Sp(3)}, \\
 \Omega_{\mathcal{N}=16}^{SO(8)} &= 2 + 4\Delta_{\mathcal{N}=16}^{SO(3)} + 2\left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^2 + \left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^3 + 3\Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(8)}, \\
 \Omega_{\mathcal{N}=16}^{SO(9)} &= 2 + 4\Delta_{\mathcal{N}=16}^{SO(3)} + 2\left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^2 + 2\Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(3)} \cdot \Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(7)} + \Delta_{\mathcal{N}=16}^{SO(9)}, \\
 \Omega_{\mathcal{N}=16}^{Sp(4)} &= 2 + 5\Delta_{\mathcal{N}=16}^{Sp(1)} + 2\left(\Delta_{\mathcal{N}=16}^{Sp(1)}\right)^2 + 2\Delta_{\mathcal{N}=16}^{Sp(2)} + \Delta_{\mathcal{N}=16}^{Sp(1)} \cdot \Delta_{\mathcal{N}=16}^{Sp(2)} + \Delta_{\mathcal{N}=16}^{Sp(3)} + \Delta_{\mathcal{N}=16}^{Sp(4)},
 \end{aligned} \tag{4.27}$$

with  $\Delta_{\mathcal{N}=16}^G$  defined in eq. (4.25).<sup>10</sup> Just as in  $SU(N)$  cases, where each fractional term involving  $\Delta_{\mathcal{N}=16}^{SU(p) \subset SU(N)}$ 's is from a specific continuum sector, we can again blame each term with  $\Delta_{\mathcal{N}=16}^{\tilde{G} \subset G}$ 's to a specific continuum sector with partial bound states, and thus read off the bona-fide Witten index as

$$\mathcal{I}_{\mathcal{N}=16}^{SO(5)=Sp(2)} = 1,$$

$$\mathcal{I}_{\mathcal{N}=16}^{G_2} = 2,$$

$$\mathcal{I}_{\mathcal{N}=16}^{SO(7)} = 1,$$

<sup>10</sup>Recall that  $\Delta^G$  depends only on the Weyl group and its action on the Cartan subalgebra of  $G$ , allowing the identities,  $\Delta^{SO(2N+1)} = \Delta^{Sp(N)}$ , as well as more trivial ones following from the equality of the Lie Algebras,  $\Delta^{SO(3)} = \Delta^{SU(2)} = \Delta^{Sp(1)}$ ,  $\Delta^{SO(5)} = \Delta^{Sp(2)}$ ,  $\Delta^{SO(4)} = (\Delta^{SU(2)})^2$ , and  $\Delta^{SO(6)} = \Delta^{SU(4)}$ .

$$\begin{aligned}
\mathcal{I}_{\mathcal{N}=16}^{\text{Sp}(3)} &= 2, \\
\mathcal{I}_{\mathcal{N}=16}^{\text{Sp}(8)} &= 2, \\
\mathcal{I}_{\mathcal{N}=16}^{\text{SO}(9)} &= 2, \\
\mathcal{I}_{\mathcal{N}=16}^{\text{Sp}(4)} &= 2,
\end{aligned} \tag{4.28}$$

which again count  $L^2$  threshold bound states.

Interestingly, the same numbers of states have been advocated in the past by Kac and Smilga [42], who inferred these numbers by studying mass-deformed versions of such quantum mechanics [46]. Of course this approach of theirs, where the ground states are classified as classical solutions, is yet another, rather heavy-handed, attempt to regulate the infrared subtleties; the result cannot be considered conclusive on its own. Nevertheless, our Witten index computation, with very different and much milder infrared regulator and in particular with the ability to keep track of various continuum contributions to  $\mathcal{I}_{\text{bulk}}^G$  sector by sector, confirms their counting for these low rank examples.

## 5 Index and rational invariant for $\mathcal{N} = 4$ quiver theories

Let us start by recalling the pure  $\text{SU}(N)$  Yang-Mills quantum mechanics with  $\mathcal{N} = 16$  supercharges, where the localization procedure gives

$$\Omega_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1; x \rightarrow 1} = \sum_{p|N} \frac{1}{p^2}. \tag{5.1}$$

As we already noted, this fractional result for the bulk part of Witten index has a well-known interpretation as in eq. (4.26), implying

$$\mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N)} = \mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1} = 1, \tag{5.2}$$

for all  $N$ . In fact, the same is true of the somewhat more trivial cases with  $\mathcal{N} = 4, 8$ , so we really have

$$\Omega_{\mathcal{N}=4,8,16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1; x \rightarrow 1} = \mathcal{I}_{\mathcal{N}=4,8,16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1} + \sum_{p|N; p>1} \frac{1}{p^2} \times \mathcal{I}_{\mathcal{N}=4,8,16}^{\text{SU}(N/p)} \Big|_{\mathbf{y} \rightarrow 1}. \tag{5.3}$$

Comparing with the actual expression of  $\Omega$ 's computed by the localization tells us immediately that

$$\begin{aligned}
\mathcal{I}_{\mathcal{N}=4,8}^{\text{SU}(N)} &= \mathcal{I}_{\mathcal{N}=4,8}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1} = 0, \\
\mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N)} &= \mathcal{I}_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1} = 1.
\end{aligned} \tag{5.4}$$



For  $\mathcal{N} = 4, 8$ , the only surviving term is  $p = N$ ,

$$\Omega_{\mathcal{N}=4,8}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1; x \rightarrow 1} = \frac{1}{N^2} \times \mathcal{I}_{\mathcal{N}=4,8}^{\text{SU}(1)} \Big|_{\mathbf{y} \rightarrow 1} = \frac{1}{N^2} \quad (5.5)$$

where the formal statement,  $\mathcal{I}_{N=4,8,16}^{\text{SU}(1)} = 1$ , refers to the fact the elementary vector multiplet is the basic BPS multiplet for each case. For  $\mathcal{N} = 4, 8$ , a similar line of thought leads us to

$$\mathcal{I}_{\mathcal{N}=4,8}^G = 0, \quad (5.6)$$

for all simple gauge groups: this was in fact one of the basic assumptions that lead us to the formulae (4.1) and (4.2), to begin with, so our confirmation of the latter by an explicit localization computation shows (5.6) self-consistently.

Such relations between  $\mathcal{I}$  and  $\Omega$  are not confined to these pure Yang-Mills quantum mechanics. While we do not understand them generally, there is at least one very general class of theories where such a pattern is quite prevalent, namely,  $\mathcal{N} = 4$  quiver quantum mechanics. For this, we first recall the so-called rational invariant for a BPS state of fixed charge  $\Gamma$ ,

$$\Omega^\Gamma(\mathbf{y}) = \sum_{p|\Gamma} \frac{\mathbf{y} - \mathbf{y}^{-1}}{p \cdot (\mathbf{y}^p - \mathbf{y}^{-p})} \times \mathcal{I}^{\Gamma/p}(\mathbf{y}^p), \quad (5.7)$$

where  $p$  is a positive integer such that  $\Gamma/p$  is a properly quantized charge (an integral rank vector). This has a known inversion formula,

$$\mathcal{I}^\Gamma(\mathbf{y}) = \sum_{p|\Gamma} \frac{\mu(p) \cdot (\mathbf{y} - \mathbf{y}^{-1})}{p \cdot (\mathbf{y}^p - \mathbf{y}^{-p})} \times \Omega^{\Gamma/p}(\mathbf{y}^p), \quad (5.8)$$

where  $\mu(p)$  is the same Möbius function that appeared briefly in section 3.4 in an entirely different context.

Note that this is precisely the type of relations we saw for  $\text{SU}(N)$  pure Yang-Mills quantum mechanics with various numbers of supersymmetries. Interestingly, these rational invariants have also surfaced more recently in the context of various wall-crossing formulae, say, either those derived from low energy dynamics of Seiberg-Witten dyons [21–23, 49] or those based on the mathematics of Donaldson-Thomas invariants and Kontsevich-Soibelman wall-crossing algebra [17–20]. Below we will review how the index of a quiver theory is organized in terms of such rational invariants, and how this leads directly to  $L^2$  Witten index that counts threshold bound states.

## 5.1 Rational invariants and primitive quivers

Recall the quantum mechanical GLSM of quiver type with  $\mathcal{N} = 4$  supersymmetries. A quiver diagram  $\mathcal{Q}$  is a pair  $(V, E)$ , where  $V$  is a finite set of vertices (or nodes) and  $E$  is a finite set of oriented edges connecting those vertices. We associate a physical theory to  $\mathcal{Q}$  as follows. Firstly, each vertex  $v_i \in V$ ,  $i = 1, \dots, n$ , is labelled by a rank  $N_i$ , representing a  $\text{U}(N_i)$  vector multiplet, and is equipped with an FI constant  $\zeta_i$  for its trace  $\text{U}(1)$ . Secondly,

each oriented edge  $e \in E$  represents a chiral multiplet in the bifundamental representation,  $(\bar{N}_{t_e}, N_{h_e})$ , where  $t_e$  and  $h_e$  are the tail and the head vertices of the edge  $e$ , respectively. One may describe  $E$  via the adjacency matrix,  $b = [b_{ij}]$ , which counts the arrows from node  $v_i$  to  $v_j$ . We will consider quivers without a flavor node, so the total gauge group is  $[\prod U(N_i)]/U(1)$  after the free  $U(1)$  is factored out.

In particular, we are confining our attention to those quivers without 1-loop and 2-loop, meaning, respectively, an edge that starts and ends at the same node and edges between a pair of nodes with mutually opposite orientations. The former,  $b_{ii} = 0$ , means that there is no adjoint chirals, and the latter,  $b_{ij} \cdot b_{ji} = 0$  for  $i \neq j$ , means any pair of chirals with mutually opposite gauge charges are taken to annihilate each other. This class of quivers arises naturally as low energy dynamics of BPS objects in  $d = 4$   $\mathcal{N} = 2$  theories [9], and has been studied quite extensively in recent years.<sup>11</sup>

For the simplest class of quivers where the rational invariant enters, consider primitive quivers  $\mathcal{Q}$  endowed with the coprime ranks  $N_i$  as well as the FI constants  $\zeta_i$ , and *without any oriented loops*. The primitivity means  $\{N_i\}$  has no nontrivial common divisor. As we will discuss later, this setting can actually be extended to general quivers with the understanding that the generalization to nonprimitive case gives the “bulk” part of the index. An Abelianization formula has been proposed [21] and proved [23] for this class of theories, which can be summarized compactly as follows [58],

$$\Omega_{\mathcal{Q}}^{\zeta} = \sum_{\mathcal{P}} \frac{1}{|\Gamma_{\mathcal{P}}|} \cdot \Omega_{\mathcal{Q}_{\mathcal{P}}}^{\zeta_{\mathcal{P}}} \cdot \Omega_{\mathcal{P}} . \quad (5.9)$$

The summation here is over the partitions  $\mathcal{P}$  of the rank vector  $(N_1, \dots, N_n)$ , specifying a partition of each rank  $N_i$  into  $l_i$  positive integers,

$$\mathcal{P} = (\{N_{1,a_1}\}_{a_1=1}^{l_1}, \dots, \{N_{n,a_n}\}_{a_n=1}^{l_n}), \quad \text{with } N_i = \sum_{a_i=1}^{l_i} N_{i,a_i} . \quad (5.10)$$

Then, to each such partition  $\mathcal{P}$  is associated an Abelian quiver,  $\mathcal{Q}_{\mathcal{P}}$ , with  $U(1)$  vertices  $v_{i,a_i}$  for  $a_i = 1, \dots, l_i$ ,  $i = 1, \dots, n$  and FI constants  $\zeta_{i,a_i} = N_{i,a_i} \zeta_i$ , collectively denoted as  $\zeta_{\mathcal{P}}$ .

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<sup>11</sup>For quivers without a loop, the problem has been worked out extensively and repeatedly. A pioneering work by Reineke [50] gave a closed-form formula for the Poincare polynomial of a Higgs vacuum moduli space, while a series of study by Manschot, Pioline, and Sen offered a very comprehensive fixed-point proposal via the Coulombic multi-center picture [21, 21]. Then, ref. [23] re-derived these formulae from physical low energy dynamics of  $d = 4$   $\mathcal{N} = 2$  dyons and also much clarified the end result mathematically. The results of the aforementioned analyses, as well as that of the wall-crossing formula due to Kontsevich and Soibelman, were subsequently shown to be equivalent among one another [24].

However, for quivers with a loop, the story turned out to be far more involved as one must consider an entirely new classes of BPS states [51, 52] that cannot be captured by the multi-center construction of BPS states. Geometric and physical meaning of such states, called “pure Higgs” or “intrinsic Higgs,” were clarified in ref. [29] and subsequently incorporated into the multi-center counting in refs. [53–55]. For quivers, the index associated with such states is called the quiver-invariant to distinguish it from Witten index. The latter changes discontinuously across marginal stability walls, while the former does not.

An orthogonal approach to the BPS spectra problem, very useful for low rank  $d = 4$   $\mathcal{N} = 2$  field theories, has also been used fruitfully, whose modern form can be traced to ref. [20]. A representative of such works can be found in refs. [56, 57].

The edge set of  $\mathcal{Q}_{\mathcal{P}}$  is specified by its adjacency, so that there are  $N_{i,a_i} N_{j,a_j} b_{ij}$  edges from the node  $v_{i,a_i}$  to  $v_{j,a_j}$ . Simply put, this induced Abelian quiver  $\mathcal{Q}_{\mathcal{P}}$  results if we partition  $N_i = \sum_{a_i} N_{i,a_i}$  and treat each  $N_{i,a_i} \times N_{i,a_i}$  block as if it is a single Abelian node. Now, the discrete group  $\Gamma_{\mathcal{P}}$  is defined by its trivial permutation action on the nodes of  $\mathcal{Q}_{\mathcal{P}}$  in the following sense: if in the partition  $N_i = \sum_{a_i} N_{i,a_i}$  we find  $N_{i,s_i}$  repeated  $d_{s_i}$  times, such that  $N_i = \sum_{s_i} d_{s_i} N_{i,s_i}$  with  $N_{i,s_i} \neq N_{i,s'_i}$  for  $s_i \neq s'_i$ , we have<sup>12</sup>

$$\Gamma_{\mathcal{P}} = \prod_i \prod_{s_i} S(d_{s_i}) . \quad (5.11)$$

The two other factors in the summand are themselves twisted partition functions on their own. The former factor,  $\Omega_{\mathcal{Q}_{\mathcal{P}}}^{\zeta_{\mathcal{P}}}$ , is the twisted partition function of the Abelian theory associated to  $\mathcal{Q}_{\mathcal{P}}$  with FI constants  $\zeta_{\mathcal{P}}$ . On the other hand, the latter factor is a product of many twisted partition functions

$$\Omega_{\mathcal{P}} = \prod_{i=1}^n \prod_{a_i=1}^{l_i} \frac{1}{N_{i,a_i}} \frac{\mathbf{y} - \mathbf{y}^{-1}}{\mathbf{y}^{N_{i,a_i}} - \mathbf{y}^{-N_{i,a_i}}} , \quad (5.12)$$

which we recognize as

$$\Omega_{\mathcal{P}} = \prod_{i=1}^n \prod_{a_i=1}^{l_i} \Omega_{\mathcal{N}=4}^{\text{SU}(N_{i,a_i})} , \quad (5.13)$$

from the previous section.

In the Coulombic picture of the wavefunctions, which is known to be effective for such loopless quivers, each summand has a clear physical interpretation as follows: a sector labeled by  $\mathcal{P}$  corresponds to the partial symmetry breaking,

$$\text{U}(N_i) \rightarrow \text{U}(1)^{l_i} \times \prod_{a_i} \text{SU}(N_{i,a_i}) , \quad (5.14)$$

where the low energy dynamics are then locally a product of  $\mathcal{Q}_{\mathcal{P}}$  with the gauge group,  $[\prod_i \text{U}(1)^{l_i}] / \text{U}(1)$ , and of many  $\mathcal{N} = 4$  pure Yang-Mills theory with the gauge groups  $\text{SU}(N_{i,a_i})$ . The twisted partition function of the former give  $\Omega_{\mathcal{Q}_{\mathcal{P}}}^{\zeta_{\mathcal{P}}}$ , while the latter gives a product of pure  $\text{SU}(N_{i,a_i})$  twisted partition functions which act like the intrinsic degeneracy attached to each node of  $\mathcal{Q}_{\mathcal{P}}$ . The discrete division by  $\Gamma_{\mathcal{P}}$  is a remnant of the original Weyl group  $\prod S(N_i)$  after the divisions due to  $S(N_{i,a_i})$  subgroups are used up in the computation of  $\Omega_{\mathcal{N}=4}^{\text{SU}(N_{i,a_i})}$ 's.

For primitive and connected quivers, the low energy dynamics is compact and the twisted partition function  $\Omega$  has to be itself the Witten index and thus integral, despite

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<sup>12</sup>For example, with a two-node quiver with the rank vector  $(N_1, N_2) = (16, 9)$  and the partition  $\mathcal{P} : (16, 9) \rightarrow (2 + 2 + 2 + 5 + 5, 1 + 1 + 1 + 1 + 1 + 4)$ , the corresponding discrete group is give as

$$\Gamma_{\mathcal{P}} = S(3) \times S(2) \times S(5) \times S(1) .$$

such a complicated Abelianization formula with fractional contributions term by term. In other words,

$$\mathcal{I}_{\mathcal{Q}} = \Omega_{\mathcal{Q}} . \quad (5.15)$$

The rational invariant enters the story in the computational middle steps via

$$\Omega_{\mathcal{N}=4}^{\text{SU}(N_{i,a_i})} = \sum_{p|N_{i,a_i}} \frac{\mathbf{y} - \mathbf{y}^{-1}}{p \cdot (\mathbf{y}^p - \mathbf{y}^{-p})} \times \mathcal{I}_{\mathcal{N}=4}^{\text{SU}(N_{i,a_i}/p)} = \frac{1}{N_{i,a_i}} \frac{\mathbf{y} - \mathbf{y}^{-1}}{\mathbf{y}^{N_{i,a_i}} - \mathbf{y}^{-N_{i,a_i}}} . \quad (5.16)$$

For nonprimitive quivers, on the other hand, the rational invariant becomes relevant for the final form of  $\Omega_{\mathcal{Q}}$  as well, since the theory then comes with gapless asymptotic directions, much similar to the pure Yang-Mills theories. We now turn to this case.

## 5.2 Rational invariants and threshold bound states

A nonprimitive quiver is a quiver whose rank vector  $\vec{N}$  is divisible by positive integer(s)  $p \geq 2$ . Such a quiver comes with gapless asymptotic directions corresponding to separating  $p$  subquivers, each denoted as  $\mathcal{Q}/p$ , with the reduced rank vector  $\vec{N}/p$ , from one another. They thus lead to a fractional continuum contribution. Our experience with  $\mathcal{N} = 4$  theories so far suggests that the twisted partition function will produce fractional results, but also that the results can be rephrased in terms of integral Witten indices of these subquivers,  $\mathcal{Q}/p$ ,

$$\Omega_{\mathcal{Q}}(\mathbf{y}) = \sum_{p|\mathcal{Q}} \frac{\mathbf{y} - \mathbf{y}^{-1}}{p \cdot (\mathbf{y}^p - \mathbf{y}^{-p})} \times \mathcal{I}_{\mathcal{Q}/p}(\mathbf{y}^p) , \quad (5.17)$$

or alternatively,

$$\mathcal{I}_{\mathcal{Q}}(\mathbf{y}) = \sum_{p|\mathcal{Q}} \frac{\mu(p) \cdot (\mathbf{y} - \mathbf{y}^{-1})}{p \cdot (\mathbf{y}^p - \mathbf{y}^{-p})} \times \Omega_{\mathcal{Q}/p}(\mathbf{y}^p) . \quad (5.18)$$

We propose that this relationship between the twisted partition function  $\Omega$ , computed by the HKY localization method, and  $\mathcal{I}$ , the true Witten index, hold for general nonprimitive quivers.<sup>13</sup> Below, we test it explicitly for some simple quivers.

The simplest class of quivers where this can be tested is the Kronecker quiver, with two nodes. With the rank vector  $(N, N)$ , the theory has the obvious gapless asymptotic directions along the Coulombic side. Denoting the quiver as  $\mathcal{Q}_b^{(N,N)}$  where  $b = b_{12}$  is the intersection number, we should have

$$\mathcal{I}_{\mathcal{Q}_b^{(N,N)}}(\mathbf{y}) = \sum_{p|N} \frac{\mu(p) \cdot (\mathbf{y} - \mathbf{y}^{-1})}{p \cdot (\mathbf{y}^p - \mathbf{y}^{-p})} \times \Omega_{\mathcal{Q}_b^{(N/p, N/p)}}(\mathbf{y}^p) , \quad (5.19)$$

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<sup>13</sup>In the context of the multi-center Coulombic index computation, where one computes in the end certain symplectic volume via a fixed point theorem, the same type of relationship has been proposed. See ref. [55] for a compact summary. Their approach deals with the problematic asymptotic direction by unphysical deformation of FI constants in the Abelianized middle steps, and also cannot compute the single center states that would be counted by the quiver invariant [29, 52]. The latter are left as unknown input data [53, 54], producing an incomplete answers when a loop is present.

which, of course, must be integral. Primitive  $N = 1$  cases reproduce well-known indices,

$$\mathcal{I}_{\mathcal{Q}_b^{(1,1)}}(\mathbf{y}) = \Omega_{\mathcal{Q}_b^{(1,1)}}(\mathbf{y}) = (-1)^{b-1} \frac{\mathbf{y}^b - \mathbf{y}^{-b}}{\mathbf{y} - \mathbf{y}^{-1}} = (-1)^{b-1} \chi_{(b-1)/2}(\mathbf{y}^2), \quad (5.20)$$

where  $\chi_s(z)$  is the usual  $SU(2)$  character of the spin  $s$  representation. The overall sign is a consequence of the chirality choice we made,  $(-1)^{2J_3}$ , and suggests that these states are  $U(1)_R$  neutral and classified via  $SU(2)_R$  multiplets.<sup>14</sup> We computed  $\Omega$ 's and thereby deduced the physical Witten index as  $\mathcal{I}$ 's

$$\begin{aligned} \mathcal{I}_{\mathcal{Q}_{b=1}^{(2,2)}}^{\zeta > 0}(\mathbf{y}) &= 0, \\ \mathcal{I}_{\mathcal{Q}_{b=2}^{(2,2)}}^{\zeta > 0}(\mathbf{y}) &= 0, \\ \mathcal{I}_{\mathcal{Q}_{b=3}^{(2,2)}}^{\zeta > 0}(\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2), \\ \mathcal{I}_{\mathcal{Q}_{b=4}^{(2,2)}}^{\zeta > 0}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2), \\ \mathcal{I}_{\mathcal{Q}_{b=5}^{(2,2)}}^{\zeta > 0}(\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2) - 2\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2), \end{aligned} \quad (5.21)$$

for  $N = 2$ , and for  $N = 3$

$$\begin{aligned} \mathcal{I}_{\mathcal{Q}_{b=1}^{(3,3)}}^{\zeta > 0}(\mathbf{y}) &= 0, \\ \mathcal{I}_{\mathcal{Q}_{b=2}^{(3,3)}}^{\zeta > 0}(\mathbf{y}) &= 0, \\ \mathcal{I}_{\mathcal{Q}_{b=3}^{(3,3)}}^{\zeta > 0}(\mathbf{y}) &= \chi_5(\mathbf{y}^2) + \chi_3(\mathbf{y}^2). \end{aligned} \quad (5.22)$$

States counted by these indices have to be at threshold. They cannot be constructed by the usual Coulombic approximation of Denef's type since this theory has a nontrivial classical moduli space even for  $\zeta < 0$ , and such a quantum bound state cannot be argued away either via the usual multi-center picture based on one-loop effective potential. Therefore, the following fact, also confirmed using the same routine as for  $\zeta > 0$  cases,

$$\mathcal{I}_{\mathcal{Q}_b^{(N,N)}}^{\zeta < 0}(\mathbf{y}) = 0, \quad (5.23)$$

is actually a nontrivial statement, confirmed by our explicit residue computations. Together, these results are consistent with the general wall-crossing algebra of Kontsevich and Soibelman.

The next simplest nonprimitive quivers are those with three nodes, say, of ranks  $(N, N, N)$ . One class is  $\mathcal{Q}_{a,b}^{(N,N,N)}$  with nonvanishing  $b_{ij}$ 's being  $(b_{12} = a, b_{23} = b)$ . This quiver has only one nontrivial chamber, and we list the Witten index for this chamber as

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<sup>14</sup>This feature is typical of loop-less quivers, and is related to the so-called no exotics conjecture [26, 59] for  $d = 4$  Seiberg-Witten theories. In the latter, our  $SU(2)_R$  would be interpreted as the spatial rotation group.

follows

$$\begin{aligned}
 \mathcal{I}_{Q_{1,1}^{(2,2,2)}}(\mathbf{y}) &= 0, \\
 \mathcal{I}_{Q_{1,2}^{(2,2,2)}}(\mathbf{y}) &= 0, \\
 \mathcal{I}_{Q_{1,3}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{1,4}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{1,5}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2) - 2\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,2}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2) - \chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,3}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{7/2}(\mathbf{y}^2) - 3\chi_{5/2}(\mathbf{y}^2) - \chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \quad (5.24)
 \end{aligned}$$

for which we also found

$$\begin{aligned}
 \mathcal{I}_{Q_{1,b}^{(1,1,1)}}(\mathbf{y}) &= (-1)^{b-1} \chi_{(b-1)/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,2}^{(1,1,1)}}(\mathbf{y}) &= +\chi_1(\mathbf{y}^2) + \chi_0(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,3}^{(1,1,1)}}(\mathbf{y}) &= -\chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \quad (5.25)
 \end{aligned}$$

along the way. Another such is  $Q_{a,b,-c}^{(N,N,N)}$  with  $(b_{12} = a, b_{23} = b, b_{13} = c)$ , where the negative sign in front of  $c$  emphasizes that one of the arrows has an opposite orientation and hence that the quiver does not have a 3-loop. In the chamber with  $\zeta_1 < 0 < \zeta_2 < \zeta_3$ , we computed the  $\Omega$ 's, from which  $\mathcal{I}$ 's can be read off as,

$$\begin{aligned}
 \mathcal{I}_{Q_{1,1,-1}^{(2,2,2)}}(\mathbf{y}) &= 0, \\
 \mathcal{I}_{Q_{2,1,-1}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2) - \chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{1,1,-2}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{5/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,2,-1}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{7/2}(\mathbf{y}^2) - 3\chi_{5/2}(\mathbf{y}^2) - \chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,1,-2}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{7/2}(\mathbf{y}^2) - 3\chi_{5/2}(\mathbf{y}^2) - 2\chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{3,1,-1}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{7/2}(\mathbf{y}^2) - 3\chi_{5/2}(\mathbf{y}^2) - 2\chi_{3/2}(\mathbf{y}^2) - 2\chi_{1/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{1,1,-3}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{9/2}(\mathbf{y}^2) - \chi_{5/2}(\mathbf{y}^2), \\
 \mathcal{I}_{Q_{2,2,-2}^{(2,2,2)}}(\mathbf{y}) &= -\chi_{13/2}(\mathbf{y}^2) - \chi_{11/2}(\mathbf{y}^2) - 4\chi_{9/2}(\mathbf{y}^2) \\
 &\quad - 3\chi_{7/2}(\mathbf{y}^2) - 4\chi_{5/2}(\mathbf{y}^2) - \chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \quad (5.26)
 \end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{\mathcal{Q}_{1,1,-1}^{(1,1,1)}}(\mathbf{y}) &= -\chi_{1/2}(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{2,1,-1}^{(1,1,1)}}(\mathbf{y}) &= +\chi_1(\mathbf{y}^2) + \chi_0(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{1,1,-2}^{(1,1,1)}}(\mathbf{y}) &= +\chi_1(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{2,2,-1}^{(1,1,1)}}(\mathbf{y}) &= -\chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{2,1,-2}^{(1,1,1)}}(\mathbf{y}) &= -\chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{3,1,-1}^{(1,1,1)}}(\mathbf{y}) &= -\chi_{3/2}(\mathbf{y}^2) - \chi_{1/2}(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{1,1,-3}^{(1,1,1)}}(\mathbf{y}) &= -\chi_{3/2}(\mathbf{y}^2), \\
\mathcal{I}_{\mathcal{Q}_{2,2,-2}^{(1,1,1)}}(\mathbf{y}) &= +\chi_2(\mathbf{y}^2) + \chi_1(\mathbf{y}^2). \tag{5.27}
\end{aligned}$$

Again, all of these are integral and sums of  $SU(2)$  characters, and agree with wall-crossing formulae, further supporting our relation (5.18).

## 6 Summary

We explored twisted partition functions  $\Omega$  of  $\mathcal{N} \geq 4$  gauged quantum mechanics with gapped and gapless asymptotic directions at quantum level, for the purpose of isolating Witten index  $\mathcal{I}$  with proper physical boundary condition. One crucial prerequisite for the localization procedure is the chemical potentials that also double as infrared regulators. We have demonstrated how the latter is often ignorant of the proper  $L^2$  boundary condition, such that the computed  $\Omega$  does not easily capture the Witten index. Of course, one should not be surprised by this at all since the boundary conditions imposed by these two are so much different from each other. While there exist formal arguments how topological quantities like index should be robust, such invariance arguments are meant to be applied to compact theories and are well-known to fail in the presence of asymptotic flat directions.

We also explored a more useful question of whether and how such  $\Omega$ 's might contain information about the true Witten indices  $\mathcal{I}$ 's. For several classes of Abelian theories, with asymptotically conical quantum moduli spaces, we have seen that different expansions of  $\Omega$  with respect to flavor chemical potentials seemingly capture various and mutually distinct cohomologies, instead of counting proper  $L^2$  states. For these classes, one does find, *a posteriori*,  $L^2$  ground state counting hidden as the common flavor-neutral sector under these various expansions, for both  $\mathcal{N} = 4$  and  $\mathcal{N} = 8$  theories. Non-Abelian theories offer further challenges as multi  $L^2$  states conspire to bring in extra integral contributions; a universal prescription for extracting  $\mathcal{I}$  from  $\Omega$  seems unlikely for the moment.

While we explored relations between  $\Omega$  and  $\mathcal{I}$  for noncompact dynamics at the level of supersymmetric quantum mechanics, observations made here are hardly confined to  $d = 1$ .



After all, the Witten index for field theories in  $d \geq 2$  dimensions often reduces to that of zero modes. Also recall that, with four supercharges and larger, the Higgs moduli space of the gauged dynamics is pretty much independent of the dimension. In particular, the related issues must be confronted for elliptic genera when the relevant superconformal field theory is not compact [60].

One additional feature for  $d = 1$  shows up when the asymptotically flat direction appears along the Coulombic side. The gap along such directions are typically controlled by the FI constant, so this type of noncompactness is quite generic since the latter can be introduced only for  $U(1)$  gauge groups. Even for theories based on  $U(N)$  gauge groups, such flat directions open up when some  $\zeta$  approaches zero, which is responsible for the wall-crossing behavior of Witten index [15] for theories that are otherwise compact. While  $\Omega$ 's for such noncompact theories are often non-integral, a systematic extraction of  $\mathcal{I}$  from  $\Omega$  is sometimes possible via the notion of rational invariants. In physics literature, glimpse of the latter was first seen some twenty years ago in the context of D-brane bound state problems [4, 39, 41], while the same notion has resurfaced more recently and more systematically in the context of the wall-crossing problem [17, 21, 23]. The upshot is that, even though a gapless Coulombic direction sounds like a worse problem than gapless Higgs directions, this is actually not the case. Since the continuum sectors contribute to  $\Omega$  in a very specific manner, we can easily identify and subtract away  $-\delta\mathcal{I}$  sitting inside  $\Omega = \mathcal{I}_{\text{bulk}}$  without much extra effort; no need to compute  $\delta\mathcal{I}$  separately, in particular.

$\Omega = \mathcal{I}_{\text{bulk}}$  for such a theory is decomposed into the integral Witten index  $\mathcal{I}$  and other fractional continuum contributions associated with certain subgroups  $\tilde{G}$  of  $G$ . The fractional part,  $-\delta\mathcal{I}$ , can be built entirely from elliptic Weyl elements of  $\tilde{G}$ 's. This map is pretty simple for theories based on  $U(N)$  or  $SU(N)$  groups, such as quiver theories, while our computation shows that a similar, more complicated, dictionary is present for other types of gauge groups as well. In this note, we have shown how this line of thinking leads to vanishing index for pure Yang-Mills quantum mechanics  $\mathcal{I}_{\mathcal{N}=4,8}^G = 0$  for all simple gauge groups  $G$  and how one can extract integral  $\mathcal{I}_{\mathcal{N}=16}^G$ 's from fractional  $\Omega_{\mathcal{N}=16}^G$ 's. For some low-rank simple groups, we actually isolated the integral Witten index  $\mathcal{I}_{\mathcal{N}=16}^G$  without a separate computation of the continuum contribution  $\delta\mathcal{I}$  as a boundary term. This is a rather remarkable property of supersymmetric gauged quantum mechanics, which may be exploited further in future.

One important and very much related matter we did not address in this note is the quiver invariant, or more generally the GLSM invariant. While we considered loop-less quivers in section 5, quivers with oriented loops are known to admit the subsector of BPS spectra that are common for all physical chambers in  $\zeta$ -space and immune to wall-crossing [29, 51, 52]. From other examples in ref. [15], it is also clear such wall-crossing-safe states would be found in many GLSM's that admit superpotentials; one obvious prototype is the GLSM that builds the familiar quintic Calabi-Yau 3-fold in the geometric phase.

These special subset of ground states can be distinguished from wall-crossing states as being  $L^2$  normalizable even at the walls of marginal stability. For this reason, the GLSM (or quiver) invariant has been proposed to be captured by the true  $L^2$  Witten index of the

theory sitting exactly at  $\zeta = 0$  [33] for all FI constants,

$$\mathcal{I}(\mathbf{y}) \Big|_{\text{GLSM Inv}} \equiv \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^F \mathbf{y}^{R+\cdots} e^{-\beta H^{\zeta=0}} \right] . \quad (6.1)$$

At such a point, all Coulombic directions open up as gapless asymptotic directions, similarly as in sections 4 and 5. In fact, the  $L^2$  Witten index for  $\mathcal{N} = 16$  gauged quantum mechanics with simple gauge groups can be thought of as the simplest prototype of this quantity, since the GLSM invariant would equal the Witten index for non-wall-crossing theories. For more general theories with  $\mathcal{N} = 4$  or less, where the GLSM invariant does not equal the Witten indices, a better way to handle the path-integral precisely at  $\zeta = 0$  is needed, to which we wish to come back in near future.

## Acknowledgments

We are indebted to Kentaro Hori and Heeyeon Kim for numerous discussions and also for collaboration at an early stage of this work. PY thanks Francesco Benini, Richard Eager, Chiung Hwang, Bumsig Kim, Sungjay Lee, Boris Pioline, and Edward Witten for useful conversations. SJL would like to thank Korea Institute for Advanced Study and Northwest University for hospitality during various stages of this work. The work of SJL is supported in part by NSF grant PHY-1417316.

## A Jeffrey-Kirwan residue

### A.1 Constructive JK residue and twisted partition function

The aim of this appendix is to summarize our algorithmic methodology for the localization computation (2.3) of twisted partition function, based on a constructive definition of the JK residue [48]. Here, we will give a detailed description of the procedure to compute the internal part contribution.<sup>15</sup> It turns out that the procedure is systematic enough to be implemented on a computer. Indeed, we developed the relevant Mathematica routines and used them extensively to obtain many of the results in this note.

To set the notations up, let us recall that the integrand  $g(u)$ , defined by eqs. (2.4), (2.5), and (2.6), have singularities along the following hyperplanes,

$$H_{\text{vector}}^\alpha = \{ u \mid \alpha \cdot u - z = 0 \}, \quad \alpha \in \Delta_G, \quad (A.1)$$

$$H_{\text{matter}}^{a,\rho} = \{ u \mid \rho \cdot u + \frac{R_a}{2} z + F_a \cdot \mu = 0 \}, \quad \rho \in \mathcal{R}_a, \quad a = 1, \dots, A. \quad (A.2)$$

We denote the set of charges for all the physical fields by

$$\mathbf{Q} \equiv \Delta_G \cup \left( \bigcup_{a=1}^A \mathcal{R}_a \right), \quad (A.3)$$

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<sup>15</sup>The asymptotic part can be taken care of along exactly the same line, except that one also needs to take into account the additional, fictitious charge  $Q_\infty = -\zeta$  for the flag combinatorics. In many cases, however, one may argue that such an asymptotic part never contributes. Indeed, we could obtain all the results in this note without ever considering the asymptotic part.

and the size of this charge set by  $N \equiv |\mathbf{Q}|$ . The index  $i$  is used to label the vector and the matter charges altogether. The charges are thus labelled as  $Q_{i=1,\dots,N}$  and the associated hyperplanes, as  $H_{i=1,\dots,N}$ . Finally, given a singularity at  $u = u_*$ , let us denote by  $\mathbf{H}_{u_*}$  the set of hyperplanes on which  $u_*$  lies,

$$\mathbf{H}_{u_*} \equiv \{ H_i \mid u_* \in H_i \}, \quad (\text{A.4})$$

and by  $\mathbf{Q}_{u_*}$  the set of their associated charges,

$$\mathbf{Q}_{u_*} \equiv \{ Q_i \mid u_* \in H_i \}. \quad (\text{A.5})$$

**Genericity and strong regularity criteria for  $\eta$ .** Before getting at the systematic approach to the twisted partition function computation, let us first clarify the genericity criterion we demand for the choice of  $\eta$  in the formula (2.3). It is known that the final outcome of the JK-Res operation, computed as the sum of JK residues over various singularities of two types, (2.8) and (2.9), does not depend on the choice of this fictitious vector  $\eta$ , given that a genericity criterion is obeyed. This demands that  $\eta$  cannot be spanned by less than  $r$  charge vectors. It should be noted that each JK residue at a given singularity may jump individually as  $\eta$  crosses a wall, while the sum (2.3) remains intact.

When we resort to the notion of flags, as is necessary when degenerate singularities are present, it turns out that such a genericity condition is not quite enough. One finds that, when  $\eta$  points along a particular integer sum of charges, the flag prescription gives too many residues. This can be seen clearly in rank two nondegenerate examples such as SU(3) pure Yang-Mills. To avoid this additional subtlety, we require the Strong Regularity condition,

$$\eta \notin \text{Cone}_{\text{sing}} \{ \sum_{i \in \pi} Q_i \mid \pi \subset \{1, \dots, N\} \}, \quad (\text{A.6})$$

where  $\text{Cone}_{\text{sing}}$  of a set of vectors denotes the union of the cones generated by any  $r - 1$  vectors in the set. The meaning of the criterion (A.6) will become clear in the context of flags below.

**JK-positive collection of the charges.** We start by collecting all the subsets of  $r$  independent charges,

$$\mathbf{Q}_{i_1, \dots, i_r} = \{ Q_{i_1}, \dots, Q_{i_r} \} \subset \mathbf{Q}, \quad (\text{A.7})$$

such that  $\text{rk}(Q_{i_1} \cdots Q_{i_r}) = r$ . We further impose that the following positivity condition is obeyed,

$$\eta \in \text{Cone}(Q_{i_1}, \dots, Q_{i_r}) \equiv \left\{ \sum_{p=1}^r a_p Q_{i_p} \mid a_p \geq 0 \right\}, \quad (\text{A.8})$$

with respect to the given choice of  $\eta$  that obeys the Strong Regularity criterion. The classification of such “JK-positive” charge collections is the starting point for the computation of the twisted partition function.

**Singularities, flags and JK residues.** Since each JK-positive collection,  $\mathbf{Q}_{i_1, \dots, i_r}$ , is linearly independent, the corresponding hyperplane collection,

$$\mathbf{H}_{i_1, \dots, i_r} = \{H_{i_1}, \dots, H_{i_r}\}, \quad (\text{A.9})$$

leads to codimension- $r$  singularities,  $u_* \in \bigcap_{p=1}^r H_{i_p}$ . In general, the  $r$  hyperplanes give rise to more than one singularities since the hyperplane equations, (A.1) and (A.2), only need to hold modulo  $2\pi i$ . Then, given the periodicity of the  $u$  variables,

$$u_k \sim u_k + 2\pi i, \quad (\text{A.10})$$

there arise finitely many discrete singularities for each JK-positive collection.

For each of the singularities  $u_*$  obtained from the collection  $\mathbf{Q}_{i_1, \dots, i_r}$ , we must proceed to construct the corresponding flags that satisfy themselves yet another positivity criterion. Note first that the singularity  $u_*$  can be degenerate so that the charge set  $\mathbf{Q}_{u_*}$  may contain more charges than the  $r$  charges  $Q_{i_1}, \dots, Q_{i_r}$ . Once  $\mathbf{Q}_{u_*}$  is found, we may forget about  $\mathbf{Q}_{i_1, \dots, i_r}$  completely and consider the set  $\mathcal{FL}(\mathbf{Q}_{u_*})$  of flags

$$\mathcal{F} = [\mathcal{F}_0 = \{0\} \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_r = \mathbb{C}^r], \quad \dim \mathcal{F}_k = k, \quad (\text{A.11})$$

such that the vector space  $\mathcal{F}_k$  at each level  $k$  is spanned by  $\{Q_{j_1}, \dots, Q_{j_k}\}$ , with  $k = 1, \dots, r$ , where the ordered set  $\mathbf{B}(\mathcal{F}) \equiv \{Q_{j_1}, \dots, Q_{j_r}\}$  is a subset of  $\mathbf{Q}_{u_*}$ . As one last piece of ingredient, we construct the vector  $\kappa_k^{\mathcal{F}}$  at each level  $k$  as,

$$\kappa_k^{\mathcal{F}} = \sum_{Q_i \in \mathbf{Q}_{u_*} \cap \mathcal{F}_k} Q_i, \quad (\text{A.12})$$

and define the sign factor,

$$\nu(\mathcal{F}) = \text{sign det}(\kappa_1^{\mathcal{F}} \cdots \kappa_r^{\mathcal{F}}). \quad (\text{A.13})$$

The JK residue at  $u_*$  is then given as,

$$\text{JK-Res}_{u=u_*}^{\eta, \mathbf{Q}_{u_*}} = \sum_{\mathcal{F} \in \mathcal{FL}^+(\mathbf{Q}_{u_*}, \eta)} \nu(\mathcal{F}) \text{Res}_{\mathcal{F}}, \quad (\text{A.14})$$

where  $\mathcal{FL}^+(\mathbf{Q}_{u_*}, \eta) = \{\mathcal{F} \in \mathcal{FL}(\mathbf{Q}_{u_*}) \mid \eta \in \text{Cone}(\kappa_1^{\mathcal{F}}, \dots, \kappa_r^{\mathcal{F}})\}$ , and the *iterated residue* associated to  $\mathcal{F}$ ,

$$\text{Res}_{\mathcal{F}}, \quad (\text{A.15})$$

is defined in terms of the basis  $\mathbf{B}(\mathcal{F})$  as follows. Given an  $r$ -form  $\omega = \omega_{1\dots r} du_1 \wedge \dots \wedge du_r$ , the following coordinate change is made,

$$\tilde{u}_k = Q_{j_k} \cdot u \quad k = 1, \dots, r, \quad (\text{A.16})$$

with respect to the basis  $\mathbf{B}(\mathcal{F})$ , and the  $r$ -form  $\omega$  is rewritten in terms of the new coordinates,

$$\omega = \tilde{\omega}_{1\dots r} d\tilde{u}_1 \wedge \dots \wedge d\tilde{u}_r. \quad (\text{A.17})$$

Then the iterated residue is defined as,

$$\text{Res}_{\mathcal{F}} \omega = \text{Res}_{\tilde{u}_r = \tilde{u}_{*r}} \cdots \text{Res}_{\tilde{u}_1 = \tilde{u}_{*1}} \tilde{\omega}_{1 \dots r}, \quad (\text{A.18})$$

where the Res at each level  $k$  on the r.h.s. is the usual single-variable residue operation, performed by regarding any other variables as a generic constant.

**Twisted Partition Function.** Once the singularities  $u_*$  of the integrand are classified, together with the associated flags  $\mathcal{FL}^+(\mathbf{Q}_{u_*}, \eta)$  with respect to a given  $\eta$ , the twisted partition function is obtained in a straightforward manner. All that is left is to compute the iterated residues following eq. (A.18), one for each pair of a singularity and a flag, and gather those residues together, taking into account the sign factor (A.13).

## A.2 Cancellation of asymptotic residue for pure Yang-Mills

Let us show that for pure Yang-Mills theories, the localization leads to no residue contributions from the asymptotic boundary of  $(\mathbb{C}^*)^r$ . Pure Yang-Mills quantum mechanics with a simple gauge group, does not have a Fayet-Iliopoulos constant that can control such an asymptotic residue via  $Q_\infty = -\zeta$ . Thus, we must go back to the basics of localization to figure out whether the asymptotic poles contribute or not.

Tracing back to localization derivation of ref. [15], we find that  $D$ -integral is performed a little differently at internal poles and at asymptotic poles. When  $\zeta$  is present and can be scaled up with  $e^2\zeta$  finite,  $D$ -integral gives a step function with the argument proportional to  $\zeta$  in such a way that, depending on JK positivity test with a given  $\eta$ , it may or may not contribute. Naively keeping  $\zeta$  finite or scaling it down to zero, this step function becomes an error function, so that, with  $\zeta = 0$  effectively, the  $D$ -integral gives 1/2 relative to when it actually contributed by passing the JK test. Now in our case, in the absence of  $\zeta$  to begin with, the end result is the same by taking the  $D$ -integral from field theory at face value and taking the principal value along  $D = 0$  pole in the integrand.

Potential contributions thus have the form,

$$\frac{1}{4\pi i} \prod_{p=1}^{r-1} \Theta(-Q_{i_p} \cdot \delta_p) \times \int_{\partial \mathbb{C}^*} du \, \mathbf{g}_{Q_{i_1}, Q_{i_2}, \dots, Q_{i_{r-1}}}(\mathbf{u}; z) \quad (\text{A.19})$$

where we have chosen some shift  $\delta_p$  of  $D$ -contours for the first  $r - 1$  integral, resulting in the step functions, while the last integral on the remaining variable, denoted as  $\mathbf{u}$ , is over the circles around the two asymptotic boundaries of the cylinder  $\mathbb{C}^*$  at  $\text{Re } \mathbf{u} = \pm\infty$ . The integrand  $\mathbf{g}$  for the  $\mathbf{u}$  integration is obtained from the original integrand  $g$  upon performing  $r - 1$  iterated residue computations. What we wish to show is that, for pure  $\mathcal{N} = 4, 8, 16$  Yang-Mills quantum mechanics, this remaining integral on the right vanishes on its own, in such a way that the complicated step functions in front becomes irrelevant.

A prototype is  $\mathcal{N} = 4$  pure  $\text{SU}(2)$  theory, for which this expression simplifies to

$$\sim \frac{1}{4\pi i} \int_{\partial \mathbb{C}^*} du \, \mathbf{g}(\mathbf{u}; z) = \frac{1}{4\pi i} \int_{\partial \mathbb{C}^*} du \, \frac{\sinh^2(\mathbf{u})}{\sinh(\mathbf{u} + z/2) \sinh(\mathbf{u} - z/2)}, \quad (\text{A.20})$$

where  $\mathbf{u}$  is the same as  $u$  as no iterated residue computation is performed in this case. The integral consists of the two boundaries  $\mathbf{u} = \pm\infty + i\phi$  but with mutually opposite orientations for  $d\phi$ . On the other hand, the integrand approaches the finite value of 1, independent of  $z$ , at both of these two boundaries, giving us

$$\frac{1}{4\pi i} (2\pi i - 2\pi i) = 0 \quad (\text{A.21})$$

It should be now clear what happens for a general gauge group, possibly with additional adjoint chirals.

For this, let us start with  $\mathcal{N} = 4$  and consider a co-dimension  $r - 1$  singularity  $l_*$  with the associated charge set  $\mathbf{Q}_{l_*}$ . Generally, the  $\{Q_{i_p}\}$  above, basis for a flag, is a subset of  $\mathbf{Q}_{l_*}$ . For a given  $l_*$ , there could be multiple contributing iterated residues but since the vanishing argument does not depend on such details, let us consider one arbitrarily choice. When the  $l_*$  is nondegenerate, of course we have  $\mathbf{Q}_{l_*} = \{Q_{i_p}\}$ .

Let us denote by  $\Delta_{l_*}^{\text{const}}$  the set of root vectors  $\alpha$  for which  $\alpha \cdot u$  is a constant over the entire line  $l_*$ . Then,  $\Delta_{l_*}^{\text{const}}$  includes  $\mathbf{Q}_{l_*}$  as a subset, responsible for the singular line  $l_*$ . We group the one-loop determinant  $g(u)$  into two factors as

$$g_{l_*}(u) \equiv \prod_{\alpha \in \Delta_{l_*}^{\text{const}}} \frac{\sinh^2(\alpha \cdot u/2)}{\sinh((\alpha \cdot u - z)/2) \sinh((\alpha \cdot u + z)/2)}, \quad (\text{A.22})$$

and

$$f_{l_*}(u) \equiv \prod_{\alpha \in \Delta \setminus \Delta_{l_*}^{\text{const}}} \frac{\sinh^2(\alpha \cdot u/2)}{\sinh((\alpha \cdot u - z)/2) \sinh((\alpha \cdot u + z)/2)}, \quad (\text{A.23})$$

so that

$$g(u) = g_{l_*}(u) \times f_{l_*}(u). \quad (\text{A.24})$$

Note that the charges involved in  $f_{l_*}(u)$  are such that  $\alpha \cdot u$  is not fixed to a number at  $l_*$ , leading to a free direction in  $u$ -space; we denote this residual direction by a vector  $s_{\text{res}}$  and parameterize the singular line  $l_*$  by the holomorphic variable  $\mathbf{u}$  as

$$u = s_{\text{res}} \mathbf{u} + u_*, \quad \mathbf{u} \in \mathbb{C}, \quad (\text{A.25})$$

where  $u_*$  is a point in  $l_*$ , chosen arbitrarily.

If only  $r - 1$  hyperplanes collide along  $l_*$ , all the poles involved in the first  $r - 1$  integrations are a simple pole and  $f_{l_*}(u)$  is merely evaluated at  $l_*$ . Thus, when it comes to the last integration,  $f_{l_*}(u)$  becomes a function only of  $\mathbf{u}$ ,

$$f_{l_*}(u) \xrightarrow{\text{restricted to } l_*} \tilde{f}_{l_*}(\mathbf{u}) := f_{l_*}(s_{\text{res}} \mathbf{u} + u_*), \quad (\text{A.26})$$

along the singular line  $l_*$ . Note in particular that

$$\tilde{f}_{l_*}(\mathbf{u} \rightarrow +\infty) = 1 = \tilde{f}_{l_*}(\mathbf{u} \rightarrow -\infty), \quad (\text{A.27})$$

at the asymptotic infinity. The final integrand  $\mathfrak{g}(\mathbf{u})$  is then the product of  $\tilde{f}_{l_*}(\mathbf{u})$  and the value of the residue from  $g_{l_*}$ , say,  $\tilde{g}_{l_*}$ . Since the latter is independent of  $\mathbf{u}$ , we thus have

$$\mathfrak{g}(\mathbf{u} \rightarrow +\infty) = \tilde{g}_{l_*} = \mathfrak{g}(\mathbf{u} \rightarrow -\infty), \quad (\text{A.28})$$

so that the residues at the two asymptotic poles sum to zero.

When more than  $r - 1$  charges collide at  $l_*$ , say  $r - 1 + n$  of them, only a slight modification occurs. Because of the presence of degenerate poles, residue operation could take further contributions from derivatives of  $g_{l_*}(u)$  and  $f_{l_*}(u)$ , so the final integrand  $\mathfrak{g}(\mathbf{u})$  takes the form,

$$\sum_{k=0}^n \sum_{a=1}^{a_k} \tilde{g}_{l_*}^{(n-k);a} \times \tilde{f}_{l_*}^{(k);a}(\mathbf{u}), \quad (\text{A.29})$$

where  $\tilde{g}_{l_*}^{(n-k);a}$  and  $\tilde{f}_{l_*}^{(k);a}(\mathbf{u})$  are appropriate results of the first  $r - 1$  residue operations. The latter is in particular obtained after taking  $k$  number of derivatives on  $f_{l_*}(u)$ , where the superscript,  $a = 1, \dots, a_k$ , labels numerous different terms with the same total number of derivatives on  $f_{l_*}(u)$ . Again, only  $\tilde{f}_{l_*}^{(k);a}(\mathbf{u})$ 's carry  $\mathbf{u}$ -dependence. Since  $f_{l_*}(u)$  is asymptotically constant up to exponentially small corrections  $\sim e^{-|c \cdot \text{Re } \mathbf{u}|}$ , their derivatives vanish asymptotically and the only surviving piece as  $\text{Re } \mathbf{u} \rightarrow \pm\infty$  is the  $k = 0$  piece,

$$\mathfrak{g}(\mathbf{u} \rightarrow \pm\infty) = \tilde{g}_{l_*}^{(n)} \times \tilde{f}_{l_*}(\mathbf{u} \rightarrow \pm\infty) = \tilde{g}_{l_*}^{(n)}, \quad (\text{A.30})$$

which again cancel against each other.

Generalization to larger supersymmetry is straightforward. The underlying reason behind this vanishing argument is that, for a given flavor and  $R$ -charge, a pair of mutually opposite gauge charges contribute together to the original integrand  $g(u)$ . When only adjoint  $\mathcal{N} = 4$  chiral multiplets are present, we can split the integrand as a product of

$$\frac{\sinh((\alpha \cdot u + qz + F \cdot \mu)/2) \sinh((- \alpha \cdot u + qz + F \cdot \mu)/2)}{\sinh((\alpha \cdot u + (q-1)z + F \cdot \mu)/2) \sinh((- \alpha \cdot u + (q-1)z + F \cdot \mu)/2)}, \quad (\text{A.31})$$

for some  $q$ 's and  $F$ 's. This approaches to 1 at the asymptotic regions of  $(\mathbb{C}^*)^r$ , regardless of  $q$  and  $F$ , which lets us proceed in the same manner as in  $\mathcal{N} = 4$  case above. Because  $\mathcal{N} = 8$  and  $\mathcal{N} = 16$  can be constructed by adding either one or three sets of adjoint chirals and a constraining superpotential appropriately, the argument for the cancelation of asymptotic residues still holds for these cases.

## B Elliptic Weyl elements

### B.1 Classical groups

An elliptic element  $w$  of Weyl group  $W$  is defined by absence of eigenvalue 1 in the canonical  $r$ -dimensional representation of  $W$  on the weight lattice. For  $\text{SU}(N)$ , the Weyl group is the permutation group  $S_N$  and the relevant representation for it is the  $(N-1)$ -dimensional irreducible one. Of elements of  $S_N$ , the only elliptic Weyl are the fully cyclic ones, say,  $(123 \cdots N)$  and the permutations thereof, where  $(\dots)$  represents the cyclic permutation.

For  $\text{SO}(2N)$ ,  $\text{SO}(2N+1)$ , and  $\text{Sp}(N)$  groups, the Weyl groups are  $S_N$  semi-direct-product with  $(Z_2)^{N-1}$ ,  $(Z_2)^N$ , and  $(Z_2)^N$ , respectively. The elements can be therefore represented as follows

$$w = (ab\dot{c}\dot{d}\dots)(klm\dot{n}\dots)\dots$$

where  $(\dots)$  again represents a cyclic permutation and dots above a number indicate a sign flip. For example  $(12\dot{3})$  represents the element,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this form, the above  $(Z_2)^{N-1}$  for  $\text{SO}(2N)$  means that the total number of sign flip has to be even. Since the determinant factorizes upon the above decomposition of  $w$ , this should be true for each cyclic component. It is fairly easy to see that this requires each cyclic component of  $w$  to have an odd number of sign flips.

Let us list the relevant conjugacy classes of classical groups, by listing typical elements in each class. We list up to rank five, except for  $\text{SU}(N)$ ,

- $\text{SU}(N)$ 

$$(123 \dots N)$$
- $\text{SO}(4)$ 

$$(\dot{1})(\dot{2})$$
- $\text{SO}(5)$  and  $\text{Sp}(2)$ 

$$(1\dot{2}), \quad (\dot{1})(\dot{2})$$
- $\text{SO}(6)$ 

$$(1\dot{2})(\dot{3})$$
- $\text{SO}(7)$  and  $\text{Sp}(3)$ 

$$(\dot{1}\dot{2}\dot{3}), \quad (12\dot{3}), \quad (1\dot{2})(\dot{3}), \quad (\dot{1})(\dot{2})(\dot{3})$$
- $\text{SO}(8)$ 

$$(\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$$
- $\text{SO}(9)$  and  $\text{Sp}(4)$ 

$$(1\dot{2}\dot{3}\dot{4}), \quad (123\dot{4}), \quad (\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (1\dot{2})(\dot{3})(\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$$
- $\text{SO}(11)$  and  $\text{Sp}(5)$ 

$$(\dot{1}\dot{2}\dot{3}\dot{4}\dot{5}), \quad (12\dot{3}\dot{4}\dot{5}), \quad (1234\dot{5}), \quad (1\dot{2}\dot{3}\dot{4})(\dot{5}), \quad (123\dot{4})(\dot{5}), \quad (\dot{1}\dot{2}\dot{3})(4\dot{5}), \quad (12\dot{3})(4\dot{5}),$$

$$(\dot{1}\dot{2}\dot{3})(\dot{4})(\dot{5}), \quad (12\dot{3})(\dot{4})(\dot{5}), \quad (1\dot{2})(3\dot{4})(\dot{5}), \quad (1\dot{2})(\dot{3})(\dot{4})(\dot{5}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})(\dot{5})$$



## B.2 Exceptional groups: $G_2$ and $F_4$

For exceptional gauge groups, the Weyl symmetries are more involved. For  $G_2$ , the dihedral group  $D_6$  with 12 elements is the Weyl group. This group is a symmetry group of 6-gon with 6 rotations and 6 reflections. Reflections cannot be elliptic, as it always leaves a direction intact. Of remaining rotations,  $Z_6$ , five proper rotations are all elliptic Weyl elements.

For  $F_4$ , the Weyl group is semidirect product of  $S_4$  which permutes the four basis,  $e_{1,2,3,4}$ ,  $(Z_2)^3$  which flips an even number of  $e_i$ 's, and  $S_3$  generated by

$$\alpha \equiv \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{B.1})$$

Alternatively, this can be thought of as a semidirect product of  $(Z_2)^4$  that flips signs of each of  $e_i$ , the permutation  $S_4$ , and the  $Z_3$  consisting of the identity and the two matrices,

$$\gamma \equiv \alpha\beta = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \gamma^2 = \gamma^{-1} = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \quad (\text{B.2})$$

In other words, the Weyl group of  $F_4$  is a semi-direct product of  $Z_3$ , generated by  $\gamma$ , with the Weyl group of  $\text{SO}(9)$ , with the cardinality  $3 \cdot 2^4 \cdot 4! = 1152$ . Since the determinant of  $\gamma$  is unit, the expression we are after for  $\Omega_{\mathcal{N}=4}^{F_4}(\mathbf{y})$  is

$$\frac{1}{1152} \left( \sum'_w \frac{1}{\text{Det}(y^{-1} - y \cdot w)} + \sum''_w \frac{1}{\text{Det}(y^{-1}\gamma - y \cdot w)} + \sum'''_w \frac{1}{\text{Det}(y^{-1}\gamma^{-1} - y \cdot w)} \right),$$

where the three sums are over subsets of  $\text{SO}(9)$  Weyl group, restricted to those  $w$ 's such that  $\text{Det}(\gamma^n - w) \neq 0$  for  $n = 0, \pm 1$ , respectively.

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## References

- [1] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys. B* **202** (1982) 253 [[INSPIRE](#)].
- [2] L. Alvarez-Gaume, *Supersymmetry And Index Theory*, in proceedings of the 1984 NATO ASI on Supersymmetry, Bonn, Germany, 20–31 Aug 1984, K. Dietz et al. eds., Plenum Press, New York U.S.A. (1985) [[INSPIRE](#)].
- [3] E. Witten, *Supersymmetry and Morse theory*, *J. Diff. Geom.* **17** (1982) 661 [[INSPIRE](#)].
- [4] G.W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [[hep-th/9803265](#)] [[INSPIRE](#)].

- [5] D. Bak, C.-k. Lee, K.-M. Lee and P. Yi, *Low-energy dynamics for 1/4 BPS dyons*, *Phys. Rev. D* **61** (2000) 025001 [[hep-th/9906119](#)] [[INSPIRE](#)].
- [6] D. Bak, K.-M. Lee and P. Yi, *Quantum 1/4 BPS dyons*, *Phys. Rev. D* **61** (2000) 045003 [[hep-th/9907090](#)] [[INSPIRE](#)].
- [7] J.P. Gauntlett, N. Kim, J. Park and P. Yi, *Monopole dynamics and BPS dyons  $N = 2$  super Yang-Mills theories*, *Phys. Rev. D* **61** (2000) 125012 [[hep-th/9912082](#)] [[INSPIRE](#)].
- [8] M. Stern and P. Yi, *Counting Yang-Mills dyons with index theorems*, *Phys. Rev. D* **62** (2000) 125006 [[hep-th/0005275](#)] [[INSPIRE](#)].
- [9] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [[hep-th/0206072](#)] [[INSPIRE](#)].
- [10] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in  $N = 2$  supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **426** (1994) 19 [Erratum *ibid.* **B 430** (1994) 485] [[hep-th/9407087](#)] [[INSPIRE](#)].
- [11] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in  $N = 2$  supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484 [[hep-th/9408099](#)] [[INSPIRE](#)].
- [12] F. Ferrari and A. Bilal, *The Strong coupling spectrum of the Seiberg-Witten theory*, *Nucl. Phys. B* **469** (1996) 387 [[hep-th/9602082](#)] [[INSPIRE](#)].
- [13] O. Bergman, *Three pronged strings and 1/4 BPS states in  $N = 4$  super Yang-Mills theory*, *Nucl. Phys. B* **525** (1998) 104 [[hep-th/9712211](#)] [[INSPIRE](#)].
- [14] K.-M. Lee and P. Yi, *Dyons in  $N = 4$  supersymmetric theories and three pronged strings*, *Phys. Rev. D* **58** (1998) 066005 [[hep-th/9804174](#)] [[INSPIRE](#)].
- [15] K. Hori, H. Kim and P. Yi, *Witten Index and Wall Crossing*, *JHEP* **01** (2015) 124 [[arXiv:1407.2567](#)] [[INSPIRE](#)].
- [16] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, Gauge Theories and Differential Geometry*, *Phys. Rept.* **66** (1980) 213 [[INSPIRE](#)].
- [17] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, [arXiv:0811.2435](#) [[INSPIRE](#)].
- [18] D. Joyce and Y. Song, *A Theory of generalized Donaldson-Thomas invariants*, [arXiv:0810.5645](#) [[INSPIRE](#)].
- [19] D. Gaiotto, G.W. Moore and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun. Math. Phys.* **299** (2010) 163 [[arXiv:0807.4723](#)] [[INSPIRE](#)].
- [20] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin Systems and the WKB Approximation*, [arXiv:0907.3987](#) [[INSPIRE](#)].
- [21] J. Manschot, B. Pioline and A. Sen, *Wall Crossing from Boltzmann Black Hole Halos*, *JHEP* **07** (2011) 059 [[arXiv:1011.1258](#)] [[INSPIRE](#)].
- [22] J. Manschot, B. Pioline and A. Sen, *A Fixed point formula for the index of multi-centered  $N = 2$  black holes*, *JHEP* **05** (2011) 057 [[arXiv:1103.1887](#)] [[INSPIRE](#)].
- [23] H. Kim, J. Park, Z. Wang and P. Yi, *Ab Initio Wall-Crossing*, *JHEP* **09** (2011) 079 [[arXiv:1107.0723](#)] [[INSPIRE](#)].

- [24] A. Sen, *Equivalence of three wall-crossing formulae*, *Commun. Num. Theor. Phys.* **6** (2012) 601 [[arXiv:1112.2515](#)] [[INSPIRE](#)].
- [25] E. Witten, *Phases of  $N = 2$  theories in two-dimensions*, *Nucl. Phys. B* **403** (1993) 159 [[hep-th/9301042](#)] [[INSPIRE](#)].
- [26] D. Gaiotto, G.W. Moore and A. Neitzke, *Framed BPS States*, *Adv. Theor. Math. Phys.* **17** (2013) 241 [[arXiv:1006.0146](#)] [[INSPIRE](#)].
- [27] C. Cordova and S.-H. Shao, *An Index Formula for Supersymmetric Quantum Mechanics*, [arXiv:1406.7853](#) [[INSPIRE](#)].
- [28] C. Hwang, J. Kim, S. Kim and J. Park, *General instanton counting and 5d SCFT*, *JHEP* **07** (2015) 063 [[arXiv:1406.6793](#)] [[INSPIRE](#)].
- [29] S.-J. Lee, Z.-L. Wang and P. Yi, *BPS States, Refined Indices and Quiver Invariants*, *JHEP* **10** (2012) 094 [[arXiv:1207.0821](#)] [[INSPIRE](#)].
- [30] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of two-dimensional  $N = 2$  gauge theories with rank-one gauge groups*, *Lett. Math. Phys.* **104** (2014) 465 [[arXiv:1305.0533](#)] [[INSPIRE](#)].
- [31] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic Genera of 2d  $\mathcal{N} = 2$  Gauge Theories*, *Commun. Math. Phys.* **333** (2015) 1241 [[arXiv:1308.4896](#)] [[INSPIRE](#)].
- [32] L.C. Jeffrey and F.C. Kirwan, *Localization for nonabelian group actions*, *Topology* **34** (1995) 291 [[arXiv:alg-geom/9307001](#)].
- [33] H. Kim, S.-J. Lee and P. Yi, *Mutation, Witten Index and Quiver Invariant*, *JHEP* **07** (2015) 093 [[arXiv:1504.00068](#)] [[INSPIRE](#)].
- [34] T. Hausel, E. Hunsicker and R. Mazzeo, *Hodge cohomology of gravitational instantons*, [math/0207169](#) [[INSPIRE](#)].
- [35] G.E. Bredon, *Topology and Geometry*, Springer (2000).
- [36] E. Witten, *Small instantons in string theory*, *Nucl. Phys. B* **460** (1996) 541 [[hep-th/9511030](#)] [[INSPIRE](#)].
- [37] H.-C. Kim, S. Kim, E. Koh, K. Lee and S. Lee, *On instantons as Kaluza-Klein modes of M5-branes*, *JHEP* **12** (2011) 031 [[arXiv:1110.2175](#)] [[INSPIRE](#)].
- [38] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys. B* **443** (1995) 85 [[hep-th/9503124](#)] [[INSPIRE](#)].
- [39] P. Yi, *Witten index and threshold bound states of D-branes*, *Nucl. Phys. B* **505** (1997) 307 [[hep-th/9704098](#)] [[INSPIRE](#)].
- [40] S. Sethi and M. Stern, *D-brane bound states redux*, *Commun. Math. Phys.* **194** (1998) 675 [[hep-th/9705046](#)] [[INSPIRE](#)].
- [41] M.B. Green and M. Gutperle, *D Particle bound states and the D instanton measure*, *JHEP* **01** (1998) 005 [[hep-th/9711107](#)] [[INSPIRE](#)].
- [42] V.G. Kac and A.V. Smilga, *Normalized vacuum states in  $N = 4$  supersymmetric Yang-Mills quantum mechanics with any gauge group*, *Nucl. Phys. B* **571** (2000) 515 [[hep-th/9908096](#)] [[INSPIRE](#)].

- [43] M. Staudacher, *Bulk Witten indices and the number of normalizable ground states in supersymmetric quantum mechanics of orthogonal, symplectic and exceptional groups*, *Phys. Lett. B* **488** (2000) 194 [[hep-th/0006234](#)] [[INSPIRE](#)].
- [44] V. Pestun,  *$N = 4$  SYM matrix integrals for almost all simple gauge groups (except  $E_7$  and  $E_8$ )*, *JHEP* **09** (2002) 012 [[hep-th/0206069](#)] [[INSPIRE](#)].
- [45] C. Hwang, private communication.
- [46] M. Porrati and A. Rozenberg, *Bound states at threshold in supersymmetric quantum mechanics*, *Nucl. Phys. B* **515** (1998) 184 [[hep-th/9708119](#)] [[INSPIRE](#)].
- [47] M. Brion and M. Vergne, *Arrangement of hyperplanes I: Rational functions and Jeffrey-Kirwan residue*, *Ann. scient. Éc. Norm. Sup.* **32** (1999) 715, [[arXiv:math/9903178](#)].
- [48] A. Szenes and M. Vergne, *Toric reduction and a conjecture of batyrev and materov*, *Inventiones Mathematicae* **158** (2004) 453 [[math/0306311](#)].
- [49] S. Lee and P. Yi, *Framed BPS States, Moduli Dynamics and Wall-Crossing*, *JHEP* **04** (2011) 098 [[arXiv:1102.1729](#)] [[INSPIRE](#)].
- [50] M. Reineke, *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli*, *Inventiones Mathematicae* **152** (2003) 349 [[math/0204059](#)].
- [51] I. Bena, M. Berkooz, J. de Boer, S. El-Showk and D. Van den Bleeken, *Scaling BPS Solutions and pure-Higgs States*, *JHEP* **11** (2012) 171 [[arXiv:1205.5023](#)] [[INSPIRE](#)].
- [52] S.-J. Lee, Z.-L. Wang and P. Yi, *Quiver Invariants from Intrinsic Higgs States*, *JHEP* **07** (2012) 169 [[arXiv:1205.6511](#)] [[INSPIRE](#)].
- [53] J. Manschot, B. Pioline and A. Sen, *From Black Holes to Quivers*, *JHEP* **11** (2012) 023 [[arXiv:1207.2230](#)] [[INSPIRE](#)].
- [54] J. Manschot, B. Pioline and A. Sen, *On the Coulomb and Higgs branch formulae for multi-centered black holes and quiver invariants*, *JHEP* **05** (2013) 166 [[arXiv:1302.5498](#)] [[INSPIRE](#)].
- [55] J. Manschot, B. Pioline and A. Sen, *The Coulomb Branch Formula for Quiver Moduli Spaces*, [[arXiv:1404.7154](#)] [[INSPIRE](#)].
- [56] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi and C. Vafa,  *$\mathcal{N} = 2$  quantum field theories and their BPS quivers*, *Adv. Theor. Math. Phys.* **18** (2014) 27 [[arXiv:1112.3984](#)] [[INSPIRE](#)].
- [57] D. Gaiotto, G.W. Moore and A. Neitzke, *Spectral networks*, *Annales Henri Poincaré* **14** (2013) 1643 [[arXiv:1204.4824](#)] [[INSPIRE](#)].
- [58] S.-J. Lee, Z.-L. Wang and P. Yi, *Abelianization of BPS Quivers and the Refined Higgs Index*, *JHEP* **02** (2014) 047 [[arXiv:1310.1265](#)] [[INSPIRE](#)].
- [59] M. Del Zotto and A. Sen, *About the Absence of Exotics and the Coulomb Branch Formula*, [[arXiv:1409.5442](#)] [[INSPIRE](#)].
- [60] J.A. Harvey, S. Lee and S. Murthy, *Elliptic genera of ALE and ALF manifolds from gauged linear  $\sigma$ -models*, *JHEP* **02** (2015) 110 [[arXiv:1406.6342](#)] [[INSPIRE](#)].